

Sparse DoA Estimation on Discretized Grids Using Hamming-Weight-Constrained Quantum Approximate Optimization Algorithm

Abstract—Estimating the Direction-of-Arrival (DoA) of multiple sources is a critical challenge in signal processing, often approached by discretizing the angular domain into a finite grid. The combinatorial nature of this problem leads to an exponential growth in complexity as the number of grid points and sources increases, making it hard for classical algorithms to solve efficiently. In this work, we frame the DoA estimation problem as a Quadratic Constrained Binary Optimization (QCBO) problem and explore the application of Hamming-Weight-Constrained-QAOA (Quantum Approximate Optimization Algorithm). Our results are promising, highlighting the effectiveness of the proposed approach.

I. INTRODUCTION

Direction-of-Arrival (DoA) estimation is a technique to estimate the direction information of several electromagnetic waves received from an array of receiving antennas [1]. Subspace-based methods like MUSIC [2] and ESPRIT [3] are widely used due to their high resolution, but admit two shortcomings: (1) they require the knowledge of the number of source, and (2) work well only when the sources are independent [1].

Sparse recovery methods use a grid of possible angles to represent where signals might be coming from. This transforms the estimation task into a sparse signal recovery problem, where the goal is to identify a small number of active sources from a large set of potential ones. Such formulations naturally fall within the framework of compressive sensing [4].

Algorithms like Sparse Iterative Covariance-based Estimation (SPICE) and Likelihood-based Estimation of Sparse parameters (LIKES) [5] are widely used for high-resolution DoA estimation [1]. Notably, they do not require prior knowledge of the number of sources or the noise variance.

However, solving the resulting optimization problem often requires searching over all possible combinations of active grid points, which becomes computationally intractable as the number of grid points increases. Additionally, as the number of sources grows, the number of feasible configurations satisfying the cardinality constraint increases exponentially. This motivates the exploration of more efficient solution techniques such as quantum algorithms.

Quantum computing is emerging as a tool for solving hard discrete optimization problems. The Quantum Approximate Optimization Algorithm (QAOA) [6] is designed for Quadratic Unconstrained Binary Optimization (QUBO) problems and offers a hybrid quantum-classical approach that can explore large solution spaces more efficiently than brute-force classical

methods. It is especially well-suited for combinatorial optimization problems because it is designed to efficiently explore large discrete solution spaces.

Many practical problems involve constraints that restrict the solution space to a subspace of the full Hilbert space — such as, to bitstrings of fixed Hamming weight. To address this, Hamming-weight-constrained QAOA is used. The motivation behind using Hamming-weight-constrained-QAOA stems from the need to estimate an optimal DoA configuration under the constraint of a fixed number of sources or DoAs—a classic Hamming weight constraint.

In this work, we reformulate the sparse DoA estimation problem as a QCBO (Quadratic Constrained Binary Optimization) problem, making it suitable for solution via Hamming-Weight-Constrained QAOA. This approach allows us to estimate the angles of arrival by searching over sparse binary supports on a discretized angle grid. We perform the search process in two stages—initially over a coarse grid, followed by a refined search within the neighboring region of the first-stage estimate to improve accuracy. We have shown through experimental evaluation that our approach performs well across a range of SNRs for different combinations of DoAs.

II. PROBLEM FORMULATION

The aim of this formulation is to transform the DoA estimation problem, into a quantum optimization problem that can be solved using QAOA.

For a given DoA θ , the steering vector $\mathbf{a}(\theta) \in \mathbb{C}^M$ is defined as:

$$\mathbf{a}(\theta) = [1 \quad e^{jkd \sin(\theta)} \quad e^{j2kd \sin(\theta)} \quad \dots \quad e^{j(M-1)kd \sin(\theta)}]^\top$$

where:

- M is the number of sensors in the array,
- d is the inter-element spacing,
- $k = \frac{2\pi}{\lambda}$ is the wavenumber,
- λ is the wavelength of the incoming signal,
- θ is the angle of arrival in radians.

We begin by constructing the array manifold matrix $\mathbf{A} \in \mathbb{C}^{M \times D}$, which is formed by horizontally stacking steering vectors corresponding to each assumed direction of arrival (DoA). Each steering vector models the phase shifts experienced by a plane wave arriving at a uniform linear array of M sensors.

$$\mathbf{A} = [\mathbf{a}(\theta_1) \quad \mathbf{a}(\theta_2) \quad \dots \quad \mathbf{a}(\theta_D)]$$

where $\mathbf{a}(\theta_i)$ is the steering vector corresponding to the direction of arrival θ_i , and $\mathcal{D}$ is the total number of sources or DoAs.

We then generate the source signal matrix $\mathbf{S} \in \mathbb{C}^{\mathcal{D} \times N}$ which contains the source signals over N snapshots, generated by drawing samples from a complex Gaussian distribution with zero mean and variance scaled by the source power P_S . With additive Gaussian noise, the received signal matrix $\mathbf{X} \in \mathbb{C}^{M \times N}$ at the sensor array is obtained by multiplying the array manifold $\mathbf{A}$ with the source signal matrix $\mathbf{S}$, and adding the noise matrix $\mathbf{W}$. This can be expressed as:

$$\mathbf{X} = \mathbf{A}\mathbf{S} + \mathbf{W}$$

where $\mathbf{W} \in \mathbb{C}^{M \times N}$ represents additive complex Gaussian noise, the noise power is calculated based on a specified signal-to-noise ratio (SNR) in decibels. The resulting noisy observation matrix $\mathbf{X}$ is used to compute the sample covariance matrix $\mathbf{R} = \mathbf{X}\mathbf{X}^H$ as the time-averaged outer product of the noisy signal. This covariance matrix serves as the primary input for DoA estimation.

Next, we perform the eigen-decomposition of the sample covariance matrix $\mathbf{R}$, which can be expressed as:

$$\mathbf{R} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^H$$

where $\mathbf{U}$ is the matrix of eigenvectors and $\mathbf{\Lambda}$ is the diagonal matrix of eigenvalues. To extract the signal subspace, we sort the eigenvalues in descending order and select the first m eigenvectors corresponding to the largest eigenvalues. These eigenvectors, stored in $\mathbf{U}_s$, represent the signal subspace, which captures the spatial directions of the incoming signals. The signal subspace is typically associated with the real signal and is used in DoA estimation.

We now use the eigenvectors $\mathbf{U}_s$ and eigenvalues $\mathbf{\Lambda}_s$ obtained from the eigen-decomposition of the covariance matrix $\mathbf{R}$ to estimate the DoAs. It involves the following expression, which relates the signal subspace to the grid of potential DoAs:

$$(\mathbf{U}_s \mathbf{U}_s^H) \text{vec}(\text{diag}(\mathbf{\Lambda}_s)) - (\mathbf{A}_{\text{grid}} \odot \mathbf{A}_{\text{grid}}) \mathbf{\Lambda} \quad (1)$$

where $\mathbf{A}_{\text{grid}}$ is the array manifold matrix for a grid of potential DoAs with K being the number of columns of $\mathbf{A}_{\text{grid}}$, i.e., the number of candidate angles in the grid, $\odot$ represents the Khatri-Rao product, which involves multiplying corresponding columns of the two matrices, and $\otimes$ represents the Kronecker product, which results in a block matrix formed by multiplying each element of one matrix by the entire second matrix.

The objective is to minimize the L_2 norm of this expression, which can be interpreted as a least-squares optimization problem, often used in subspace-based DoA estimation methods. By minimizing this L_2 norm, we can estimate the DoAs of the incoming signals.

III. HAMILTONIAN CONSTRUCTION

A. Formulation of the Objective Function

The DoA estimation problem can be expressed as minimizing the mismatch between the observed signal covariance and

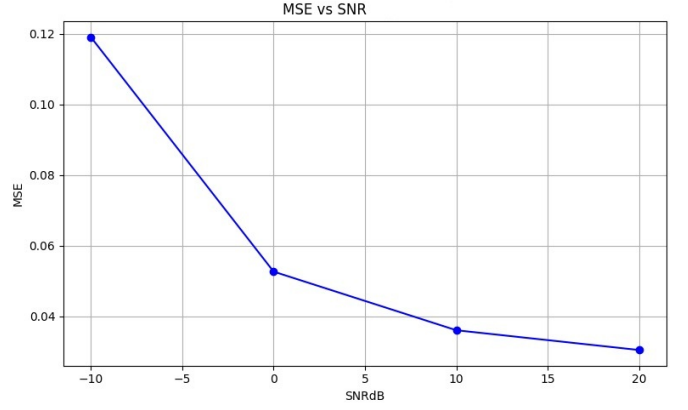


Fig. 1: MSE versus SNR plot for DoA estimation at true angles of [3, 9]. The plot illustrates how estimation accuracy improves (lower MSE) with increasing SNR for the specified DoA values.

a model constructed from a discrete grid of candidate angles. Specifically, we seek a sparse binary vector $\mathbf{\Lambda} \in \{0, 1\}^K$ that selects which grid angles contribute to the signal.

Mathematically, this is formulated as:

$$\begin{aligned} & \min_{\mathbf{\Lambda} \in \{0, 1\}^K} \left\| (\mathbf{U}_s \otimes \mathbf{U}_s^H) \text{vec}(\text{diag}(\mathbf{\Lambda}_s)) - (\mathbf{A}_{\text{grid}} \odot \mathbf{A}_{\text{grid}}) \mathbf{\Lambda} \right\|^2 \\ & \text{subject to} \quad \sum_i \Lambda_i = \mathcal{D} \end{aligned} \quad (2)$$

B. Formulation as QCBO

To solve the binary optimization problem introduced above using quantum algorithms such as QAOA, the DoA estimation task must be reformulated as a Quadratic Constrained Binary Optimization (QCBO) problem.

Let us define:

- $\mathbf{B} = (\mathbf{U}_s \otimes \mathbf{U}_s^H) \text{vec}(\text{diag}(\mathbf{\Lambda}_s))$
- $\mathbf{C} = \mathbf{A}_{\text{grid}} \odot \mathbf{A}_{\text{grid}}$

Thus, the cost term becomes:

$$\begin{aligned} & \left\| \mathbf{B} - \mathbf{C}\mathbf{\Lambda} \right\|^2 = \mathbf{\Lambda}^T \mathbf{C}^T \mathbf{C} \mathbf{\Lambda} - 2\mathbf{B}^T \mathbf{C} \mathbf{\Lambda} + \mathbf{B}^T \mathbf{B} \\ & \text{subject to} \quad \sum_i \Lambda_i = \mathcal{D} \end{aligned} \quad (3)$$

This can be expanded as:

$$\begin{aligned} & \sum_{i=1}^K \sum_{j=1}^K \Lambda_i \Lambda_j \left(\sum_{p=1}^{M^2} \mathbf{C}_{pi} \mathbf{C}_{pj} \right) - 2 \sum_{i=1}^K \Lambda_i \left(\sum_{p=1}^{M^2} \mathbf{C}_{pi} \mathbf{B}_p \right) + \sum_{i=1}^{M^2} \mathbf{B}_i^2 \\ & \text{subject to} \quad \sum_i \Lambda_i = \mathcal{D} \end{aligned} \quad (4)$$

The constraint is a Hamming constraint, this structure naturally aligns with and motivates the use of Hamming-weight-constrained QAOA for solving the problem. We do not need to convert the QCBO problem to QUBO (Quadratic Unconstrained Binary Optimization) form as it directly handles the Hamming weight constraint inherent in the problem, which eliminates the need for additional transformations.

True DOA	SNRdB	Estimated DOA (QAOA)	Error:MSE (QAOA)
[5]	-30	[4.826]	8.22932
	-20	[4.95]	6.85298
	-10	[5.532]	0.35180
	0	[5.505]	0.32410
	10	[4.997]	0.00019
[3,9]	-20	[3.252, 8.485]	4.729
	-10	[3.039, 8.839]	0.125
	0	[2.848, 8.964]	0.06006
	10	[2.934, 8.945]	0.064
	20	[2.989, 9.05]	0.0207

TABLE I: Comparison of DoA Estimation and Mean Squared Error (MSE) at Various SNR Levels for different DoAs.

In the above final optimization formulation the last term can be ignored, since it is a constant. This forms the final Hamiltonian.

To derive Ising Hamiltonian, we substitute $\Lambda_r = \frac{1-Z_r}{2}$, where Z_r is the Pauli-Z operator acting on qubit r . This defines the problem instance provided to the Hamming-weight-constrained QAOA algorithm.

IV. METHODOLOGY

In this study, we investigate the performance of Direction of Arrival (DoA) estimation using a grid-based approach, based on the Hamiltonian designed in Equation 4. To estimate the DoAs, the angular space is discretized into a uniform grid of fixed number of points spanning the potential angle range of interest.

In the first stage, the coarse grid serves as the initial estimate by identifying the peaks corresponding to the most likely DoAs.

In the second stage, we refine each coarse estimate by constructing a finer sub-grid between the two neighboring grid points of the selected peak. Specifically, if the coarse estimate falls on grid point θ_i , we define a local search interval from θ_{i-1} to θ_{i+1} . The estimation procedure is repeated on this finer grid to get better estimate of DoAs.

For each SNR level, the estimation process is repeated 100 times, each time using a different random noise signal. This helps capture the variability in performance due to noise. After each run, the estimated DoAs are compared to the true values and MSE is calculated. The MSE is then averaged over the 100 runs for each SNR value to assess overall estimation accuracy.

A. Framework for Hamming-weight-constrained QAOA

For Hamming-weight-constrained QAOA, the Mixer Hamiltonian of the form $H_M = \sum_{j,k} H_{j,k}$, where $H_{j,k} = X_j X_k + Y_j Y_k$ is used, which has been shown to be useful in the Adiabatic Quantum Optimization setting for a variety of optimization problems with hard and soft constraints [7]; such mixers restrict the mixing to the feasible subspace defined by the hard constraints [8]. XY -Hamiltonians serve as effective mixing operators when dealing with hard constraints [9].

Unlike the traditional QAOA algorithm which uses an equal superposition state as the initial state, the Hamming-Weight-Constrained-QAOA selects a Dicke state as the initial state

which restrict the quantum evolution to a subspace of states with a specific Hamming weight.

Dicke State is defined as an equal superposition of all R qubit states $|x\rangle$ having Hamming Weight $wt(x) = Q$:

$$|D_Q^R\rangle = R C_Q^{-\frac{1}{2}} \sum_{x \in \{0,1\}^R, wt(x)=Q} |x\rangle \quad (5)$$

When the total number of grid points is K , irrespective of the number of DoAs $\mathcal{D}$, the traditional QAOA algorithm starts from the equal superposition state $(\frac{1}{2^K})^{\frac{1}{2}} \sum_{x \in \{0,1\}^K} |x\rangle$. But in Hamming-weight-constrained QAOA, the initial state is influenced by the number of DoAs $\mathcal{D}$, which is $|D_{\mathcal{D}}^K\rangle$. Although quantum algorithms exist for preparing Dicke states [10], we refrain from using that circuit and instead directly initialize the qubits to the desired state.

V. RESULTS AND SIMULATIONS

In this work, we determine the angles of arrival by identifying sparse binary support vectors across a predefined, discretized angle grid. The Hamiltonian formulated in equation (4) is used in the QAOA algorithm to compute the optimum state.

For specific initial parameters, Table I shows the performance of DoA estimation. We show the Mean Squared Error (MSE) of QAOA-based methods across various Signal-to-Noise Ratio (SNR) levels for various DoAs.

In our simulations, the number of discretized angle grid points K was fixed to 20, corresponding to 20 equidistant points between 0° and 10° . This choice was motivated by the qubit constraint of current quantum simulation capabilities, as each grid point maps to a qubit in the QAOA framework. The number of sensors was fixed at 10, and the SNR in dB was varied to assess performance under different noise conditions. The inter-element spacing d was set to 5 units, and the wavelength was set to 1.

The first stage performs a coarse DoA estimation over these 20 uniform grid points. In the second stage, a finer grid is constructed between neighboring points around each selected angle to refine the estimates.

We have used *Adam'sOptimizer* [11] as the classical optimizer in QAOA for tuning the circuit parameters. We choose depth of the circuit p to be 1, i.e, the quantum circuit consists of one alternating layers: one layer for the mixed Hamiltonian followed by one layer for the cost Hamiltonian. We used Qiskit [12] for the simulations of both algorithms. To execute the quantum circuit, QAOA leveraged the *default.qubit* backend [13].

Results, as shown in Table I, demonstrate that the QAOA-based method, for specific initial parameters, achieves quite low MSE. This holds true across different SNRs and angles, showing that QAOA can reliably estimate the DoA with high accuracy.

VI. CONCLUSION

In this work, we presented a two-stage grid-based approach for DoA estimation using quantum-classical hybrid algorithm QAOA. The combinatorial nature of the problem motivates the use of QAOA for efficient exploration of the solution space. This quantum-based DoA estimation results were found to have low MSE consistently, validating the effectiveness of the proposed Hamiltonian-based approach. As future work, we can extend our DoA simulations to accommodate larger sensor arrays, more interfering sources, and more sophisticated Hamiltonian formulations.

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