

Exploring Optimal and Suboptimal Beamforming Solutions Using Weighted Subspace-Search-VQE

Abstract—This work explores the application of a Quantum algorithm, namely Weighted Subspace-Search-Variational-Quantum-Eigensolver (SSVQE), a variant of the traditional VQE algorithm, to predict optimal as well as sub-optimal sparse array configurations for the problem of beamforming. The optimization formulation is designed to maximize the Signal-to-Interference-plus-Noise Ratio (SINR), which is then reformulated into Quadratic Unconstrained Binary Optimization (QUBO) representation to serve as the Hamiltonian for the SSVQE algorithm. The results obtained are presented along with classically obtained SINR plots to facilitate a comparison of the sparse array configurations. Our findings indicate that the output from the quantum algorithm aligns closely with the results obtained from the classical method.

I. INTRODUCTION

Beamforming is a signal processing technique used in sensor vectors to direct the transmission or reception of signals[1]. Recent advances in mobile communications, usage of large arrays, high-frequency sensors, near-field signal recovery, and smart radio environments open up interesting and novel signal processing problems in beamforming. These applications are driving the need for higher robustness, flexible deployment, and low complexity in beamforming algorithms and an emphasis on advanced signal processing that should be tailored for emerging application-specific requirements[2].

Literature has explored various techniques for optimizing sparse array configurations in beamforming, demonstrating that strategic placement of antennas can significantly enhance signal quality, reduce interference, and improve computational efficiency in diverse applications. The configuration of a sparse array involves selecting positions for a limited number of antennas from a potentially vast set of locations. As the number of possible placements increases exponentially with the number of antennas and available positions, this creates a complex combinatorial problem. Quantum computers promise to solve these problems by offering solutions that may be exponentially faster than the classical equivalent[3]. To reduce the substantial hardware requirements associated with quantum computing and to fully harness the capabilities of Noisy Intermediate-Scale Quantum (NISQ) devices, the Variational Quantum Eigensolver (VQE) algorithm was introduced[4].

The VQE[5] acts as an effective algorithm to find the lowest Eigenvalue of a quantum observable. In particular, it can compute the ground-state energy of a Hamiltonian[6].

To extend the framework to excited states, the Subspace-Search-Variational-Quantum-Eigensolver (SSVQE)[7] was

proposed which searches a low-energy subspace by supplying orthogonal input states to the variational ansatz and relies on the unitarity of transformations to ensure the orthogonality of the output states. From the entire space, the focus is restricted to a subspace defined by $(r + 1)$ Eigenvectors where the r^{th} excited state is obtained as the highest-energy state in the low-energy subspace.

In this work, we aim to address the beamforming problem through the application of the weighted SSVQE algorithm. In critical applications (e.g., communication systems, medical devices), having a reliable second-best configuration can serve as a backup if the optimal configuration fails or encounters issues. Reducing reliance on a single “best” configuration can help maintain operational capability. Thus, we are interested in exploring a diverse range of configurations through SSVQE that can adapt to varying conditions and minimize the risk of system failures.

We begin by providing a brief overview of the SSVQE algorithm. Next, we derive the optimization formulation for beamforming with the goal of maximizing the Signal-to-Interference-plus-Noise Ratio (SINR). Once the Hamiltonian is established, we incorporate it into the algorithm to obtain both optimal and suboptimal solutions. Finally, we present the results in graphical form and compare them with classical results.

II. WEIGHTED SSVQE

Expectation measurement involves projecting the state of the system onto the Eigenvectors of the operator, thereby revealing the probabilities of obtaining specific outcomes. This principle is used in the traditional VQE algorithm which employs projecting the state of the system onto the Eigenvectors of the Hamiltonian to find the smallest Eigenvalue and the corresponding Eigenvector of the given Hamiltonian.

The SSVQE[7] algorithm involves minimizing

$$\sum_{d=0}^r \langle \psi_d | U^\dagger(\theta) H U(\theta) | \psi_d \rangle \quad (1)$$

where, $U(\theta)$ acts as the trainable ansatz, H is the involved Hamiltonian, ψ_d are the initial states, $0 \leq r \leq 2^N - 1$, where N is total number of qubits. Equation (1) basically restricts the entire space to a subspace spanned by $(r + 1)$ Eigenvectors of the Hamiltonian H .

The above minimization expression can be further be simplified as:

$$\sum_{d=0}^r \langle U(\theta) \psi_d | H | U(\theta) \psi_d \rangle \quad (2)$$

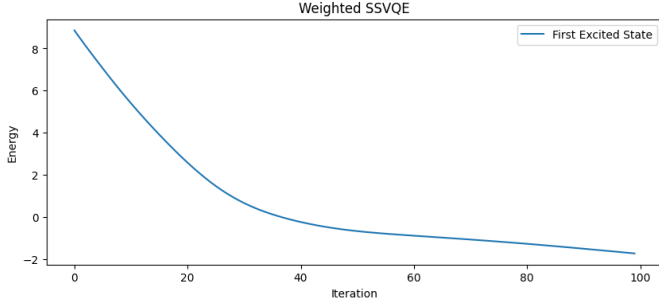


Fig. 1: Training curve for the SSVQE algorithm with parameters $N = 5, K = 4, r = 1, \theta_d = 48^\circ, \phi_i = 37^\circ$. The curve shows that it has reached a saturation for the first excited state or second-best optimal configuration.

which signifies that once we reach the optimal ansatz parameter θ^* , we can interpret that $U(\theta^*)$ is the unitary which maps the space of $\{|\psi_d\rangle\}_{d=0}^r$ to $\{|E_d\rangle\}_{d=0}^r$ with $|E_d\rangle$ being the Eigenstates of H such that $E_i \geq E_j$ for $i \geq j$. So, the entire space is now restricted to the subspace defined by $\{|E_d\rangle\}_{d=0}^r$. The Eigenvector corresponding to the highest Eigenvalue in this subspace is indeed the Eigenvector corresponding to $(r+1)^{th}$ Eigenvalue.

The weighted SSVQE introduces a weight $w \in (0, 1)$ in order to distinguish this Eigenvector. It minimizes:

$$w\langle\psi_r|U^\dagger(\theta)HU(\theta)|\psi_r\rangle + \sum_{d=0}^{r-1}\langle\psi_d|U^\dagger(\theta)HU(\theta)|\psi_d\rangle \quad (3)$$

This would guarantee, $U(\theta^*)$ maps $|\psi_r\rangle$ to $|E_r\rangle$ and the rest of the input states to the remaining Eigenstates of lower Eigenvalues. Thus, $|E_r\rangle$ becomes the Eigenvector corresponding to highest Eigenvalue in the subspace, also, it is the Eigenvector corresponding to $(r+1)^{th}$ Eigenvalue.

In the beamforming problem, given N sensors, there are 2^N sparse array possible configurations, ranging from 0 to $2^N - 1$. We use weighted SSVQE algorithm to define a subspace of $(r+1)$ Eigenvectors of the Hamiltonian by optimizing the ansatz. We then find the $(r+1)^{th}$ Eigenstate where $r = 0$ represents the best configuration and $r = 1$ corresponds to the second-best configuration and so on. This method helps in determining the $(r+1)^{th}$ best optimal configuration of sparse arrays for beamforming, thus, we use this method to determine sub-optimal sparse array configurations.

III. MINIMUM VARIANCE DISTORTIONLESS BEAMFORMER

Consider N omnidirectional antennas arranged in uniform linear array (ULA) with inter-element spacing of $p_nd, p_n \in \mathbb{N}$, with $d = \lambda/2$ and λ being the operating wavelength. Consider this array receiving signals, with a desired signal arriving from direction θ_d and q other interfering signals from various

directions $\{\phi_1, \phi_2, \dots, \phi_q\}$ [8]. The corresponding steering vectors are given by,

$$\mathbf{u}_a = [e^{jkp_1d \cos \theta_d} \quad e^{jkp_2d \cos \theta_d} \quad \dots \quad e^{jkp_Nd \cos \theta_d}]^T \quad (4)$$

$$\mathbf{v}_i = [e^{jkp_1d \cos \phi_i} \quad e^{jkp_2d \cos \phi_i} \quad \dots \quad e^{jkp_Nd \cos \phi_i}]^T \quad (5)$$

where, $i = 1, 2, \dots, q$ and the wavenumber is denoted by $k = 2\pi/\lambda$, The received signal at time instant t is given by,

$$\mathbf{x}(t) = \mathbf{u}_a s(t) + \mathbf{V} \mathbf{v}(t) + \mathbf{n}(t)$$

where $\mathbf{V} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_q]$ is the interference array manifold matrix with a full column rank, $s(t) \in \mathbb{C}$ denotes the desired signal waveform with power σ_s^2 , $\mathbf{v}(t) \in \mathbb{C}^q$ is the vector of q different statistically independent interfering signals with the covariance matrix $\mathbf{R}_v$ and $\mathbf{n}(t) \in \mathbb{C}^N$ is the received noise vector with power σ_n^2 . Noise is assumed to be Gaussian in nature.

The input SNR, defined as the ratio between the power of the signal to the power of the noise, is given as: $SNR_i = \frac{\sigma_s^2}{\sigma_n^2}$, and output SINR of MVDR beamformer, which measures the strength of the desired signal compared to the combined effects of interference and noise at the output, is given as, $SINR_o = \sigma_s^2 \mathbf{u}_a^H \mathbf{R}_n^{-1} \mathbf{u}_a$.

IV. OPTIMIZATION FORMULATION

The SINR of MVDR beamformer is given by:

$$SINR = \sigma_s^2 \mathbf{u}_a^H \mathbf{R}_n^{-1} \mathbf{u}_a \quad (6)$$

where $\mathbf{R}_n = \mathbf{V} \mathbf{R}_v \mathbf{V}^H + \sigma_n^2 \mathbf{I}$.

The constraint being $\mathbf{1}^T \mathbf{z} = K$.

Substituting $\mathbf{R}_n$ in $SINR$,

$$\begin{aligned} SINR &= \sigma_s^2 \mathbf{u}_a^H [\mathbf{V} \mathbf{R}_v \mathbf{V}^H + \sigma_n^2 \mathbf{I}]^{-1} \mathbf{u}_a \\ &= \frac{\sigma_s^2}{\sigma_n^2} \mathbf{u}_a^H [\mathbf{I} + \frac{\mathbf{V} \mathbf{R}_v \mathbf{V}^H}{\sigma_n^2}]^{-1} \mathbf{u}_a \end{aligned} \quad (7)$$

We restrict ourselves to one interfering signal.

For $q = 1$, $\mathbf{R}_v = \sigma_v^2$, i.e. variance of the signal $\mathbf{v}(t)$. Thus, Equation (7) becomes,

$$SINR = \frac{\sigma_s^2}{\sigma_n^2} \mathbf{u}_a^H [\mathbf{I} + \frac{\sigma_v^2}{\sigma_n^2} \mathbf{V} \mathbf{V}^H]^{-1} \mathbf{u}_a \quad (8)$$

We utilize Woodbury Matrix Identity [9] expressed as $(\mathbf{I} + \mathbf{U} \mathbf{V})^{-1} = \mathbf{I} - \mathbf{U}(\mathbf{I} + \mathbf{V} \mathbf{U})^{-1} \mathbf{V}$ to expand equation (8) as :

$$SINR = \frac{\sigma_s^2}{\sigma_n^2} \mathbf{u}_a^H \left[\mathbf{I} - \frac{\sigma_v^2}{\sigma_n^2} \mathbf{V} (\mathbf{I} + \frac{\mathbf{V}^H \sigma_v^2 \mathbf{V}}{\sigma_n^2})^{-1} \mathbf{V}^H \right] \mathbf{u}_a \quad (9)$$

Since we assumed $q = 1$, so the term $\mathbf{V}^H \mathbf{V}$ is consequently of dimension 1×1 .

Thus,

$$SINR = \frac{\sigma_s^2}{\sigma_n^2} \left[\mathbf{u}_a^H \mathbf{u}_a - \frac{\sigma_v^2}{\sigma_n^2} \frac{\mathbf{u}_a^H \mathbf{V} \mathbf{V}^H \mathbf{u}_a}{1 + \frac{\sigma_v^2}{\sigma_n^2} \mathbf{V}^H \mathbf{V}} \right] \quad (10)$$

N: Total sensors	K: Active Sensors	SINR plots	Histogram obtained from traditional VQE	r^{th} state	Histogram for $(r+1)^{th}$ optimum state in SSVQE	Optimum state
5	3			0		11011
				1		11001
				2		10101
5	4			0		11011
				1		10111

TABLE I: Results showing histogram plots depicting the distribution of $(r+1)^{th}$ optimum states after the execution of the SSVQE algorithm for different values of N and K obtained for certain initial parameters. θ_d is set at 48° with SNR 0 dB while ϕ_i is set at 37° with INR 5 dB. We also show the classically obtained SINR values for possible configurations along with traditional VQE plots for ease of comparison. The sensor configurations are also shown where '1' indicates the sensor is active, '0' indicates it is turned off.

Thus, maximizing $SINR$ would be equivalent to:

$$\max \frac{\sigma_s^2}{\sigma_n^2} \left[\mathbf{u}_a^H \text{diag}(\mathbf{z}) \mathbf{u}_a - \frac{\sigma_V^2}{\sigma_n^2} \frac{\mathbf{u}_a^H \text{diag}(\mathbf{z}) \mathbf{V} \mathbf{V}^H \text{diag}(\mathbf{z}) \mathbf{u}_a}{1 + \frac{\sigma_V^2}{\sigma_n^2} \mathbf{V}^H \text{diag}(\mathbf{z}) \mathbf{V}} \right] \quad (11)$$

We can substitute $\mathbf{u}_a^H \text{diag}(\mathbf{z}) \mathbf{u}_a = K$. Thus the optimization problem becomes:

$$\max \frac{\sigma_s^2}{\sigma_n^2} \left[K - \frac{\sigma_V^2}{\sigma_n^2} \frac{\mathbf{u}_a^H \text{diag}(\mathbf{z}) \mathbf{V} \mathbf{V}^H \text{diag}(\mathbf{z}) \mathbf{u}_a}{1 + \frac{\sigma_V^2}{\sigma_n^2} K} \right] \quad (12)$$

Thus, $\max SINR$ becomes equivalent to :

$$\min \mathbf{u}_a^H \text{diag}(\mathbf{z}) \mathbf{V} \mathbf{V}^H \text{diag}(\mathbf{z}) \mathbf{u}_a \quad (13)$$

V. UTILIZING WEIGHTED SSVQE FOR BEAMFORMING

Again, we are interested not only in finding the optimal sparse array configuration but also in identifying other viable solutions that are good or slightly less optimal. While the traditional VQE algorithm does find out the ground state configuration, SSVQE works on a subspace of the Eigenvectors of the Hamiltonian, thus aiding in predicting other effective solutions.

The Hamiltonian to be solved using SSVQE should be expressed as a linear combination of Pauli terms. Thus, we now express Equation (13) as Quadratic Unconstrained Binary Optimization (QUBO) problem, which is further reformulated as Ising Hamiltonian.

Formulation as QUBO : To derive the QUBO formulation, we rewrite equation (13) as:

$$\sum_{i=1}^N \sum_{j=1}^N z_i z_j (u_{a_i}^* v_i v_j^* u_{a_j}) \quad (14)$$

We use the Lagrange multiplier to make it unconstrained problem. Incorporating the constraint as quadratic penalization factor in equation (14), we get the optimization formulation as:

$$\sum_{i=1}^N \sum_{j=1}^N z_i z_j (u_{a_i}^* v_i v_j^* u_{a_j} + \mu) - 2K\mu \sum_{i=1}^N z_i + \text{constant} \quad (15)$$

To derive Ising Hamiltonian from Equation (15), we substitute $z_p = \frac{1-Z_p}{2}$ where Z_p is the Pauli-Z operator acting on qubit p , which is then used in the SSVQE algorithm. We can ignore the constant term in the above Hamiltonian since it does not have any effect on the relative energy differences among quantum states.

The Ising Hamiltonian derived from Equation (15) acts as the Hamiltonian H of Equation (1). The parameter r is determined by the level of optimality required for the configuration, i.e. the variable r is set according to the $(r+1)^{th}$ suboptimal solution required. This, in turn, will define a subspace characterized by $(r+1)$ Eigenvectors of Hamiltonian H .

A randomly initialized ansatz $U(\theta)$ is trained to obtain optimized parameters θ^* such that $U(\theta^*)|\psi_r\rangle$ can accurately map to the Eigenvector representing the required $(r+1)^{th}$ optimal configuration.

For instance, when interested in finding the second-best configuration of sparse array whose optimization formulation is represented by a Hamiltonian H , we set $r = 1$ and train the arbitrarily configured ansatz to obtain a Unitary function which would map $|1\rangle$ to $|E_1\rangle$ where $|E_1\rangle$ is the first excited-state of the Hamiltonian H , also the highest-energy state in the low-energy subspace spanned by $\{|E_0\rangle, |E_1\rangle\}$ such that $|E_0\rangle$ and $|E_1\rangle$ are two Eigenstates of H having least Eigenvalues with $E_1 > E_0$. $|E_1\rangle$ becomes our desired second-best configuration.

VI. RESULTS AND SIMULATIONS

The formulation obtained from Equation (15) acts as the Hamiltonian for which we aim to find the $(r+1)^{th}$ optimized state, where r lies in the range 0 to $2^N - 1$, N : total number of qubits involved.

Table I presents the results of our analysis. For a combination of K active sensors out of a total of N sensors, simulated on N qubits, the table displays the $(r+1)^{th}$ suboptimal configuration as obtained from the Weighted SSVQE algorithm. Additionally, for the same values of N and K , we present a

histogram derived from the traditional VQE algorithm, along with the corresponding SINR values obtained through classical methods for ease of comparison.

The SINR plots, which aid in better visualization of optimal sparse array configurations, show $\binom{N}{K}$ combinations on the x -axis, while the corresponding output SINR values computed by equation (6) along y -axis. The configuration corresponding to highest output SINR is the best optimal configuration for that particular set of parameters.

We selected $N = 5$ and examined two cases: for $K = 3$, we considered values of $r = 0, 1, 2$; for $K = 4$, the values of r were 0 and 1. While maintaining the desired signal angle θ_d at 48° with a signal-to-noise ratio (SNR) of 0 dB, the angle of the interfering signal ϕ_i is set at 37° with an interference-to-noise ratio (INR) of 5 dB.

For weighted SSVQE, we choose PennyLane's [10] Adam Optimizer [11] with stepsize = 0.1 as the classical optimizer. We use PennyLane's *StronglyEntanglingLayer* [12] as the ansatz, utilizing randomly initialized parameters within the range of 0 to 2π . For the results shown in Table I we use 5 qubits since total number of sensors $N = 5$ with *default.qubit* device. We randomly initialize the weight parameter w within the range of 0 to 1.

The training curve for the SSVQE algorithm is shown in Figure 1 for a particular set of parameters of $N = 5, K = 4, r = 1, \theta_d = 48^\circ, \phi_i = 37^\circ$. The training curves indicate that the model have reached a point of optimal performance with minimal fluctuations in loss.

The results presented in Table I indicate that the SSVQE histogram plots align closely with the SINR plots. Although the highest probability state matches with the highest SINR, it is unable to distinguish between degenerate states (or quasi-degenerate). This limitation has also been observed previously in the context of the traditional VQE algorithm [13].

The SSVQE histogram obtained for $r = 0$ represents the ground state energy, corresponding to the most optimized configuration, which matches with the traditional VQE outputs. Consequently, $r = 1$ indicates the second-best configuration, and this pattern continues for higher values of r . The highest value in the SSVQE histogram always corresponds to the $(r+1)^{th}$ best configuration. The results highlight the capability of quantum algorithms to address intricate optimization problems in beamforming.

VII. CONCLUSION

We utilized the SSVQE algorithm to identify optimal sparse array configurations for beamforming, targeting those that are optimal or close to optimal. The results when compared to classically obtained SINR plots and traditional VQE histograms were in accordance. Minimizing dependence on a single "best" configuration can enhance operational resilience, and the algorithm performs effectively in this context. This shows the potential of quantum algorithms to solve complex optimization problems.

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