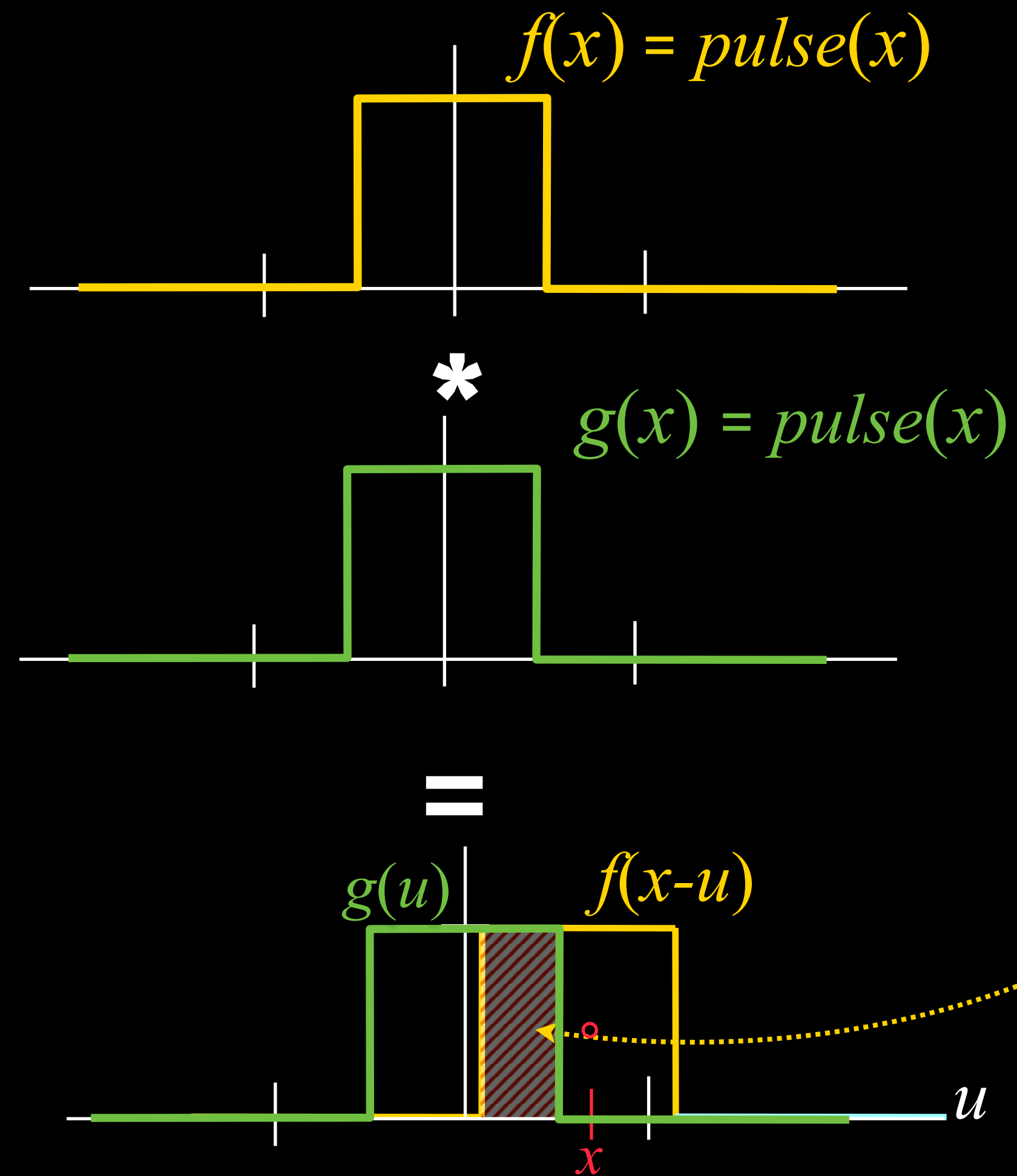
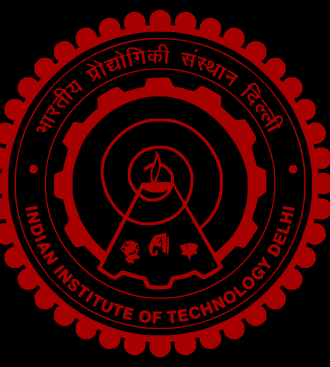


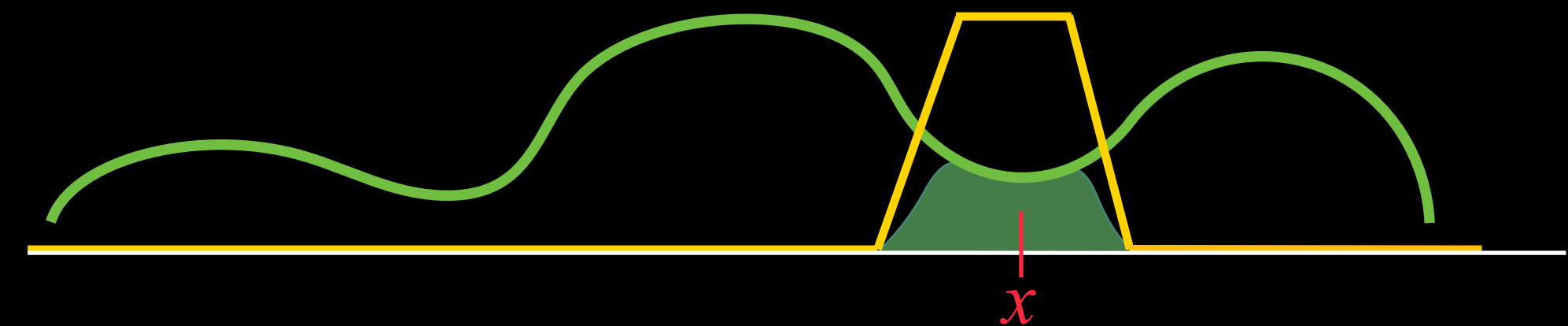
Convolution



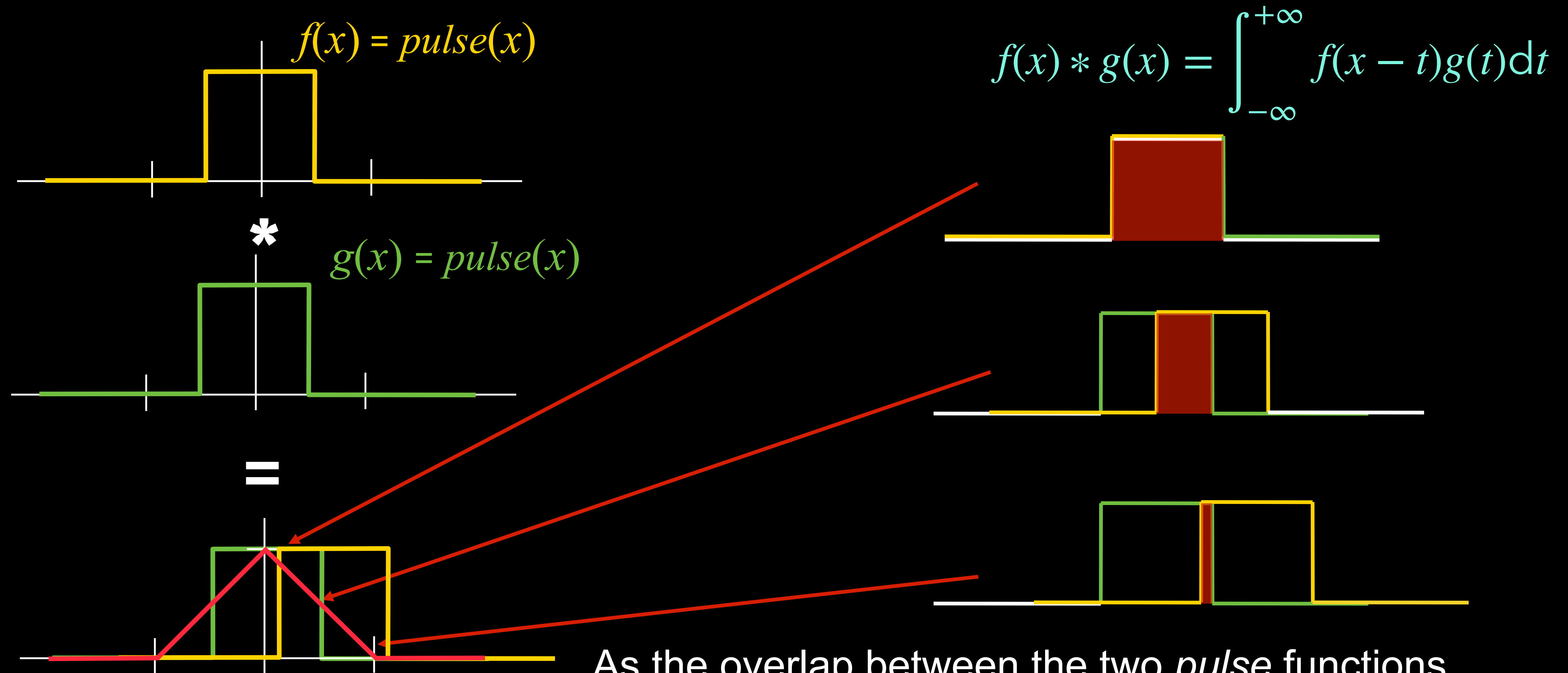
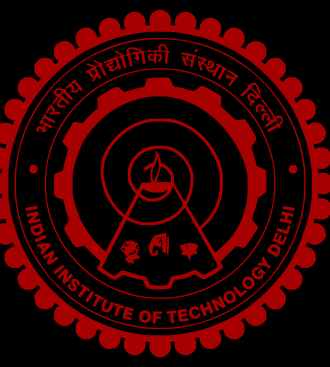
$$f(x) * g(x) = \int_{-\infty}^{+\infty} f(x-u)g(u)du$$

shaded area

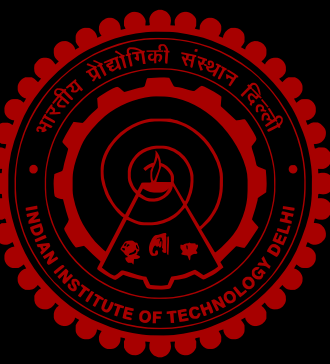
Weighted average of $g(u)$,
weights given by $f(x-u)$



Convolution



As the overlap between the two *pulse* functions reduces, the area under the product curve decreases



Convolution Theorem

If F and G be the transforms of f and g :

$$f(x) * g(x) \longleftrightarrow F(\mu) \times G(\mu)$$

$$f(x) \times g(x) \longleftrightarrow F(\mu) * G(\mu)$$

Fourier Series and Transform

Review

- Series:

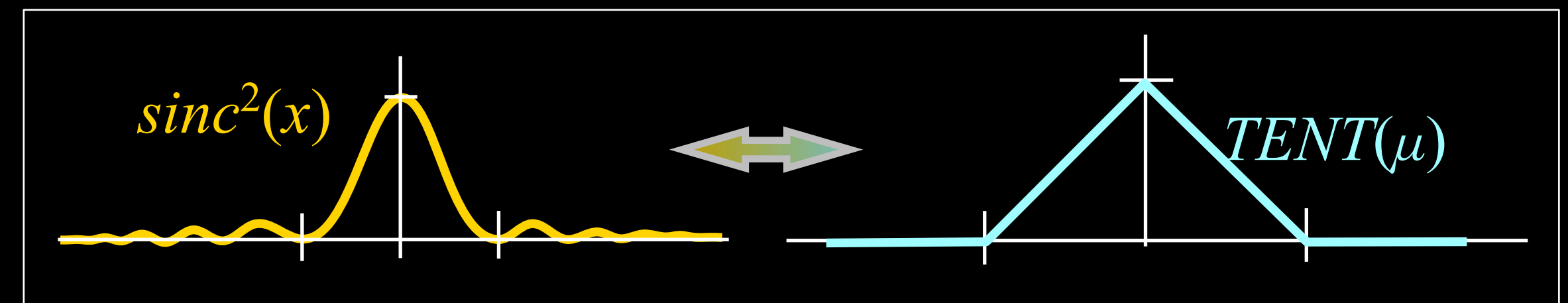
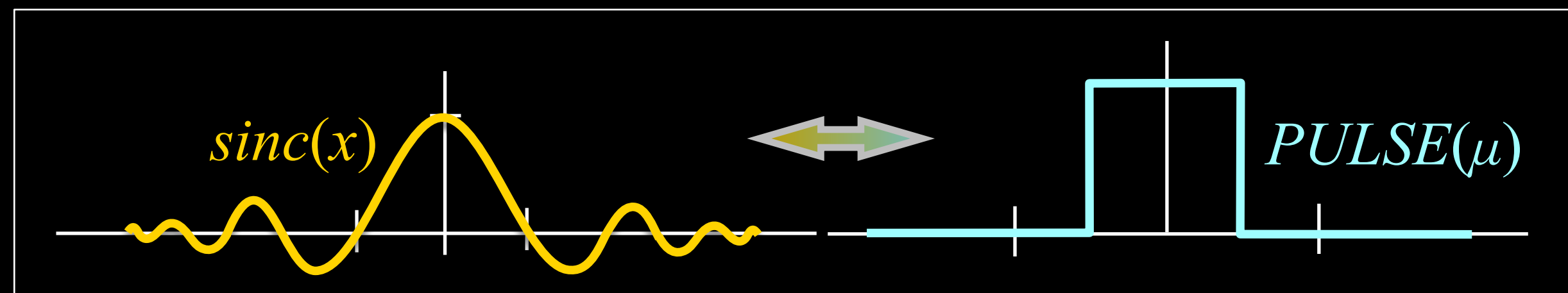
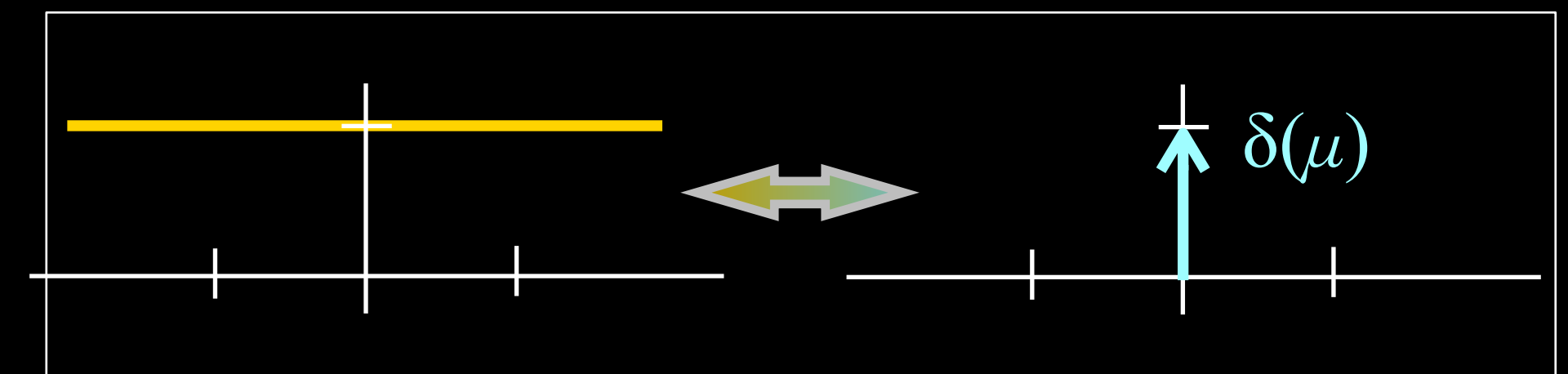
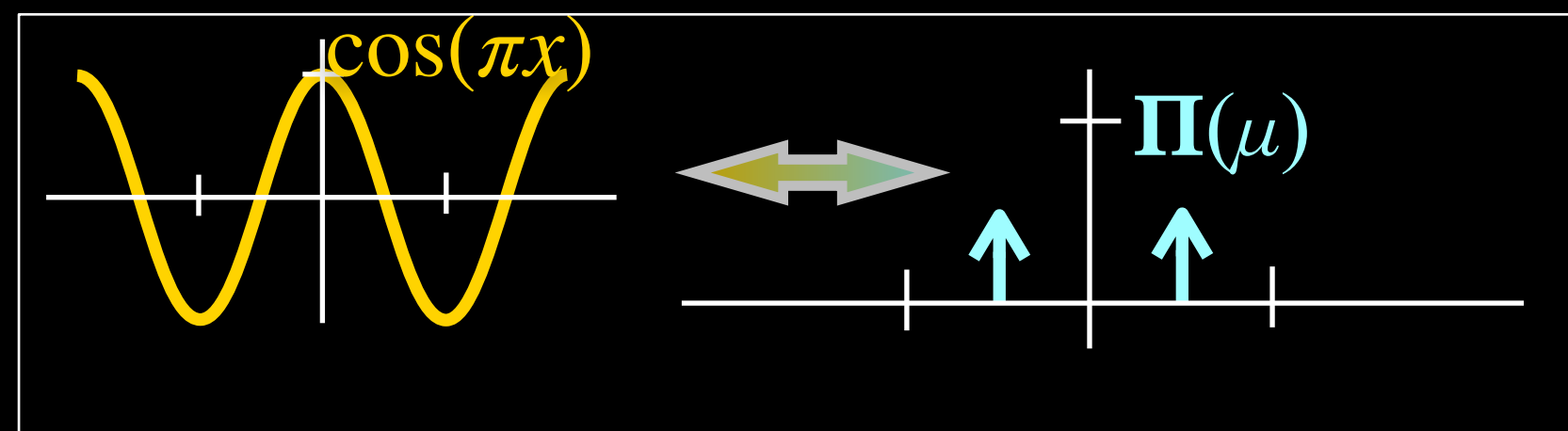
$$f(x) = \sum_{n=-\infty}^{+\infty} F_n e^{i2\pi \frac{nx}{T}}, \text{ where } F_n = \int_0^T f(x) e^{-i2\pi \frac{nx}{T}} dx$$

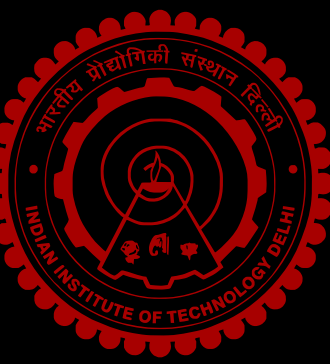
- Transform:

$$f(x) = \int_{-\infty}^{+\infty} F(\mu) e^{i2\pi \mu x} d\mu, \text{ where } F(\mu) = \int_{-\infty}^{+\infty} f(x) e^{-i2\pi \mu x} dx$$

Frequency/Fourier domain representation

- Transform pairs:





Useful Fourier Transform Properties

- Reciprocity, $f \Leftrightarrow F$: If $F(\mu)$ is a transform of $f(x)$, then $F(x)$ is a transform of $f(-\mu)$
- Inverse Scaling: If $f(x) \Leftrightarrow F(\mu)$, then $f(ax) \Leftrightarrow \frac{1}{a} F\left(\frac{\mu}{a}\right)$
- Translation: If $f(x) \Leftrightarrow F(\mu)$, then $f(x - x_0) \Leftrightarrow F(\mu) e^{-2\pi i \mu x_0}$
- Linearity: If $f \Leftrightarrow F$ and $g \Leftrightarrow G$, then $af + bg \Leftrightarrow aF + bG$
- Symmetry: If f is real and even, then F is real and even [Even f : $f(-x) = f(x)$]
 - If f is real and odd, then F is imaginary and odd

2D Fourier Transform

$$F(\mu, \nu) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) e^{-i2\pi\mu x} e^{-i2\pi\nu y} dx dy$$

Separable

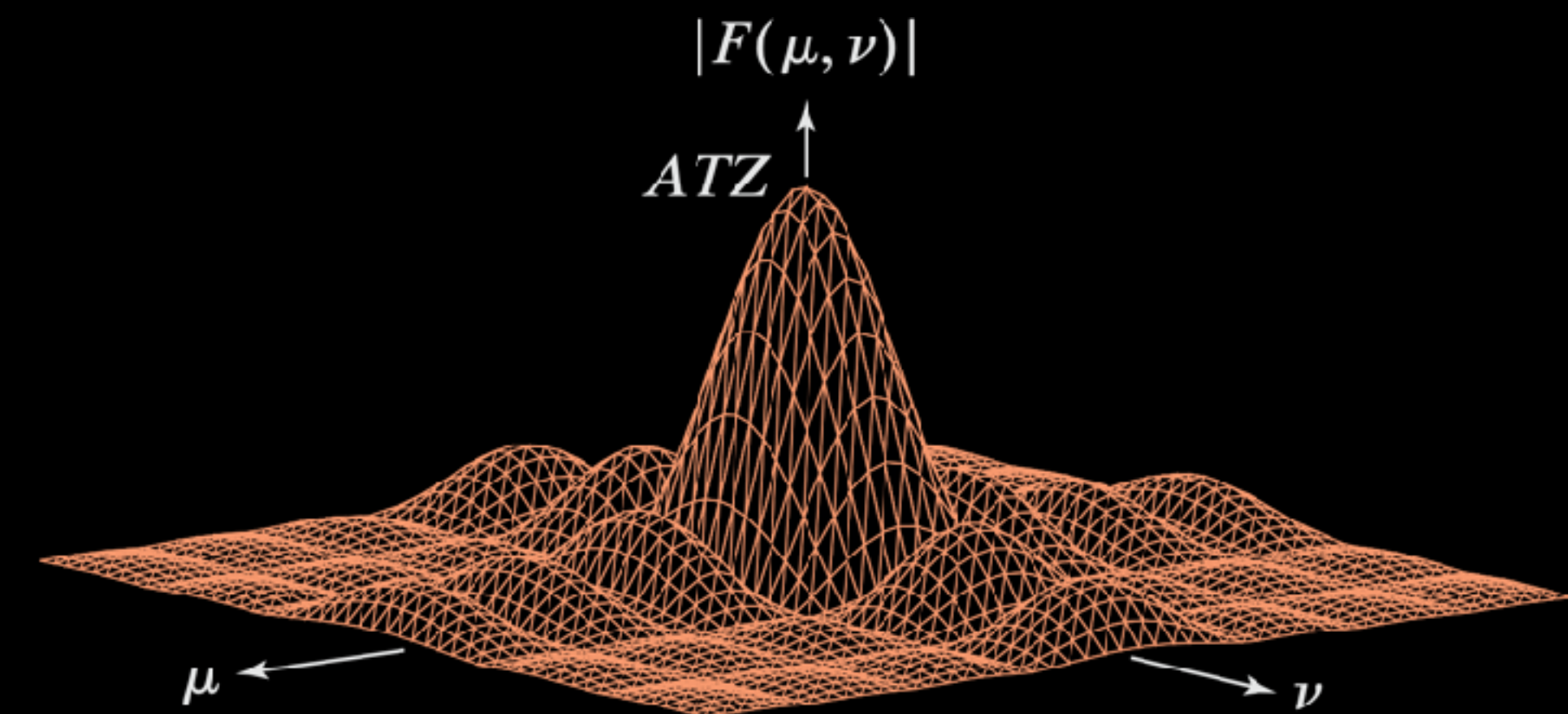
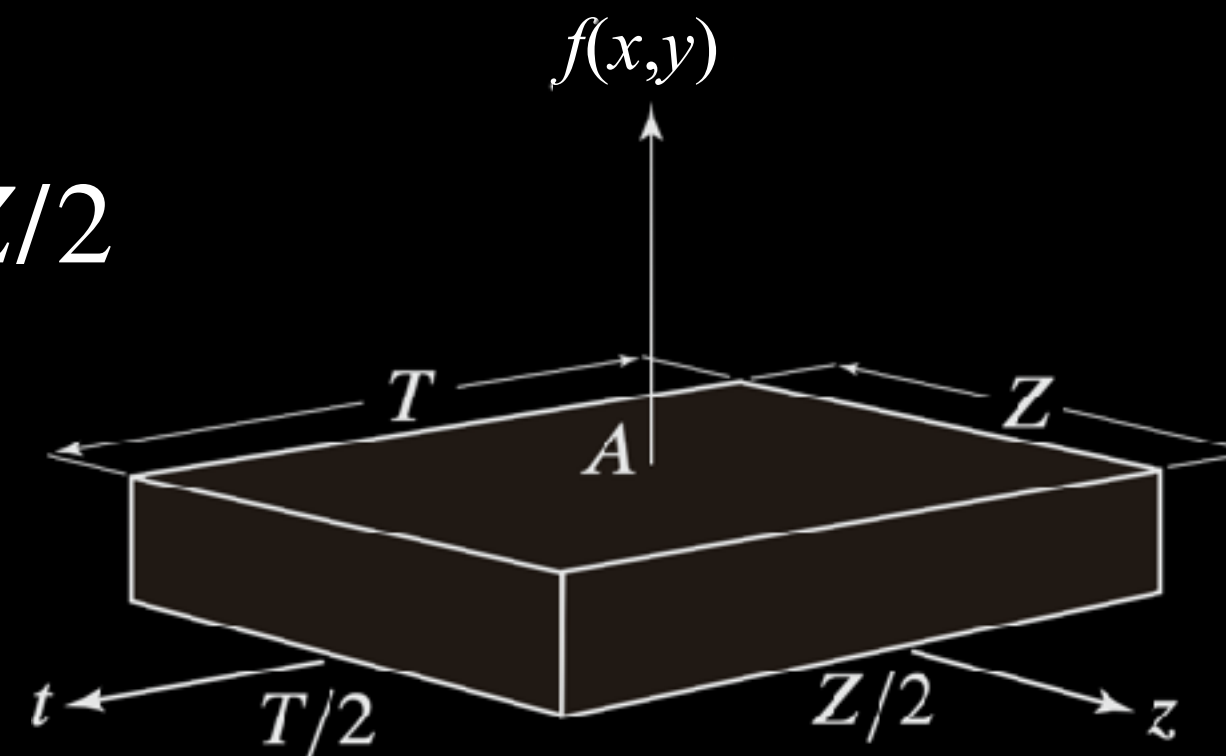
$$F(\mu, \nu) = \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(x, y) e^{-i2\pi\mu x} dx \right) e^{-i2\pi\nu y} dy$$

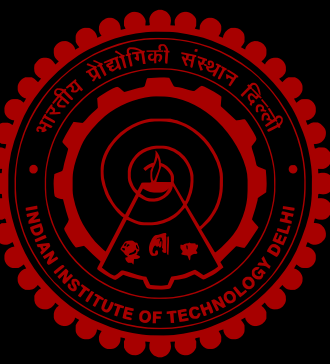
$$f(x, y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(\mu, \nu) e^{i2\pi\mu x} e^{i2\pi\nu y} d\mu d\nu$$

Example: 2D box

$$f(x, y) = \begin{cases} A & |x| \leq T/2, |y| \leq Z/2 \\ 0 & \text{otherwise} \end{cases}$$

$$F(\mu, \nu) = ATZ \operatorname{sinc}(\mu T) \operatorname{sinc}(\nu Z)$$



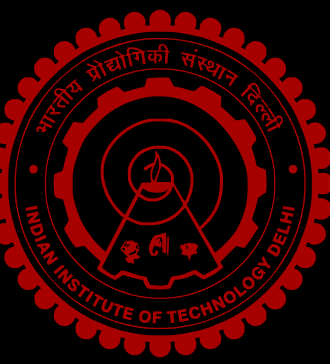


Discrete Fourier Transform

$$F(\mu) = \int_{-\infty}^{+\infty} f(x) e^{-i2\pi\mu x} dx \longrightarrow F(\mu_k) = \sum_{n=-\infty}^{\infty} f(x_n) e^{-i2\pi\mu_k x_n}$$

- f is specified as f_n or $f[n]$; uniformly placed values: $\dots -2\Delta x, \Delta x, 0, \Delta x, 2\Delta x \dots$
 - i.e., $x_n = n\Delta x$ for integer n (finitely many in practice, $n \in [0:M-1]$)
 - If we consider its periodic extension by repeating it on both sides (and with period M)
 - there are only finite Fourier coefficients at $\mu_k = \frac{k}{M\Delta x}$
- For the Fourier transform of the original signal, $k \in [0:M-1]$

$$F[k] = \sum_{n=0}^{M-1} f[n] e^{-i2\pi \frac{k}{M\Delta x} n\Delta x} = \boxed{\sum_{n=0}^{M-1} f[n] e^{-i2\pi \frac{kn}{M}}}$$

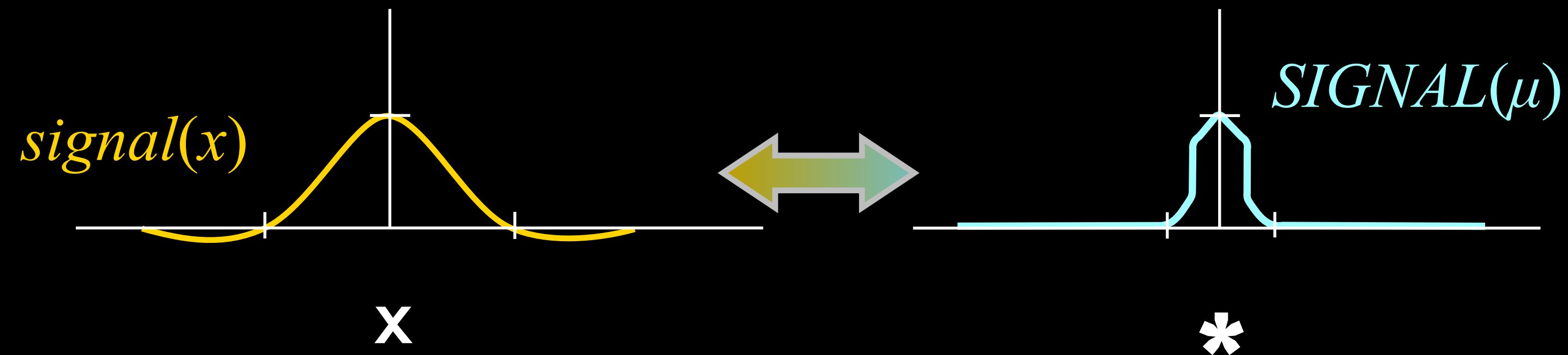
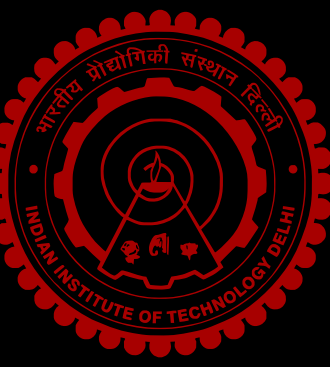


Discrete Convolution

$$f(x) * g(x) = \int_{-\infty}^{+\infty} f(x-t)g(t)dt \longrightarrow (f * g)[n] = \sum_{m=-\infty}^{+\infty} f[n-m]g[m]$$

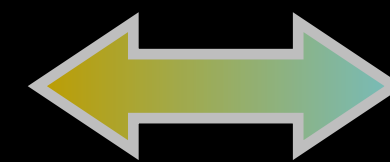
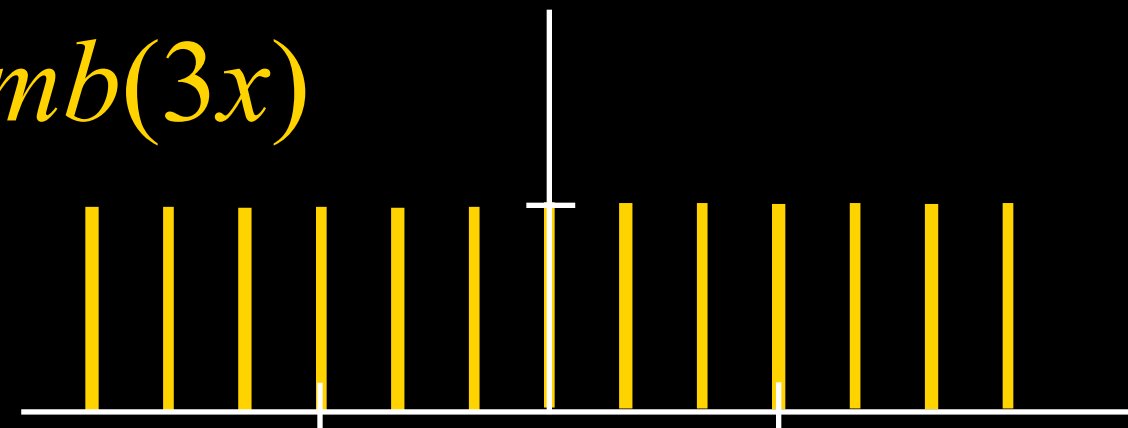
- 2D:
$$(f * h)[j, k] = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m, n] h[j-m, k-n]$$
- Circular convolution:
$$(f * h)[j, k] = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m, n] h[(j-m) \% M, (k-n) \% N]$$

Sampling

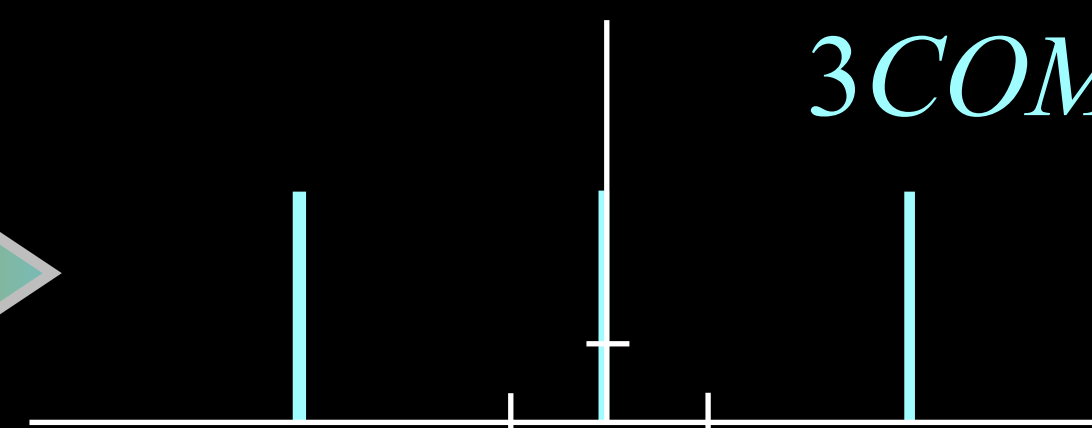


Impulse
Train:

$comb(3x)$



$3COMB(\mu/3)$

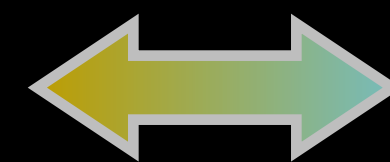
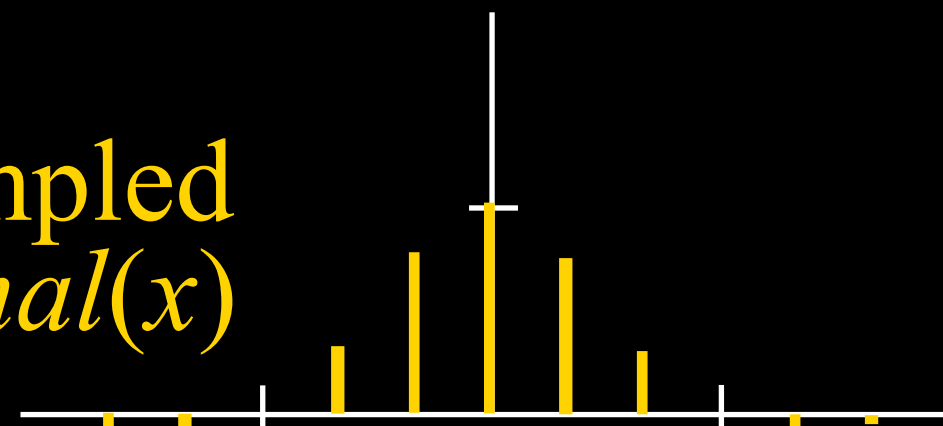


FT of $comb$ is an impulse
train with inverse period

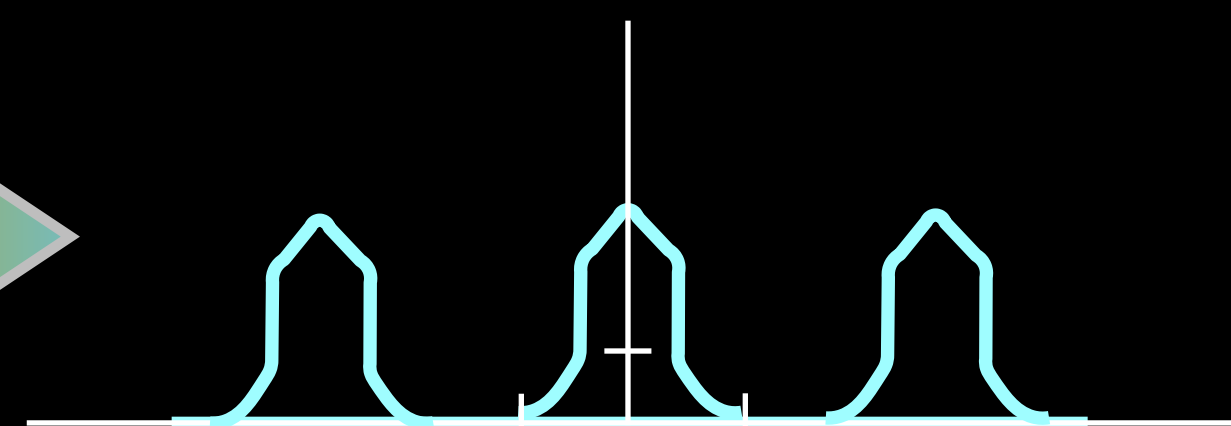
=

$$\sum_{n=-\infty}^{\infty} f_n \delta[k - n]$$

Sampled
 $signal(x)$



=



$$\sum_{n=-\infty}^{\infty} f(n\Delta x) \delta(x - n\Delta x)$$

$f(x)$

$F(\mu)$

$$\delta(x) = \begin{cases} \infty & \text{at } x = 0 \\ 0, & \text{elsewhere} \end{cases}$$

$$\delta[x] = \begin{cases} 1 & \text{at } x = 0 \\ 0, & \text{elsewhere} \end{cases}$$