



Recovering Parameters of Model

Model belongs to a known class

- If noisy data fits a model W , *e.g.*, $O = W(P)$. Say, W is represented by $M \in \mathbb{R}^m$
 O and P are vectors
(M symbolizes W)

i.e., $O_i \approx W(P_i)$

- Best model fit:

$$\arg \min_{M \in \mathbb{R}^m} E = \sum_i \text{Diff}(O_i, W(P_i)), \text{ e.g., } \arg \min_{M \in \mathbb{R}^m} \sum_i \|O_i - W(P_i)\| \text{ (L}_2\text{: least squares)}$$

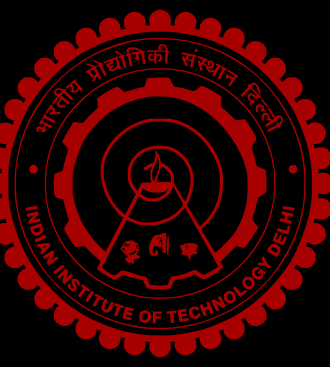
- To minimize E , solve:

If M denotes matrix multiplication, the least squares solution can be found analytically

$$\frac{\partial E}{\partial M_j} = 0 \text{ for all components } M_j \text{ of } M$$

Similarly, for restricted DOF matrix transformations

In general, use Levenberg-Marquardt method (bundle adjustment)

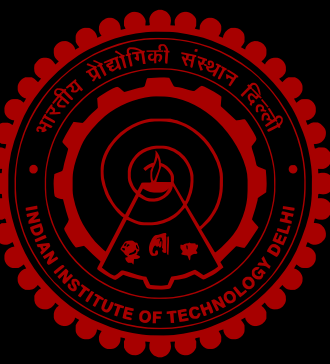


Levenberg-Marquardt

n data points

$$E(M) = \sum_{i=1}^n \|r_i(M)\|^2, \text{ where residual } r_i(M) = O_i - W(P_i) : O \text{ is } d\text{-dimensional, } M \text{ is } m\text{-dimensional}$$

$$\frac{\partial E}{\partial M_k} = 0$$



Levenberg-Marquardt

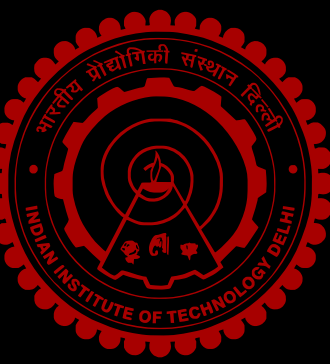
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Solve for change to M at each step: $(J^T J + \lambda \text{diag}(J^T J)) \Delta M = -J^T r$
(Linear system)

~Hessian H gives
~local curvature

Jacobian: Derivative
of every residual wrt
 M at every data point



Levenberg-Marquardt

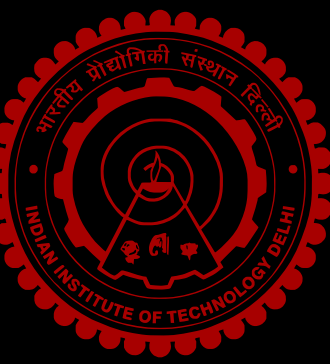
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Solve for change to M at each step: $(J^T J + \lambda \text{diag}(J^T J)) \Delta M = -J^T r$ (Linear system)

λ : Damping parameter

$J^T J$: ~Hessian H gives ~local curvature

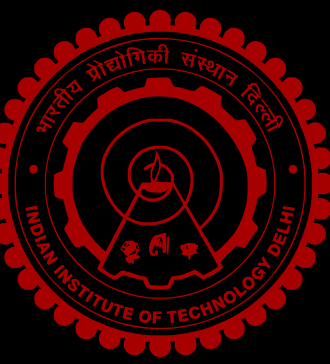
Jacobian: Derivative of every residual wrt M at every data point

$r = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$

Component-wise, $r_i = \begin{bmatrix} r_{i1}(M) \\ \vdots \\ r_{id}(M) \end{bmatrix}$

$$\frac{\partial r_{ij}}{\partial M_k} = - \frac{\partial W(P_i)_j}{\partial M_k}$$

$$\text{Jacobian } J(r) = \begin{bmatrix} J_1(r) \\ \vdots \\ J_n(r) \end{bmatrix}_{n \times d \times m}, \text{ where } J_i(r) = \begin{bmatrix} \frac{\partial r_{i1}}{\partial M_1} & \cdots & \frac{\partial r_{i1}}{\partial M_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_{id}}{\partial M_1} & \cdots & \frac{\partial r_{id}}{\partial M_m} \end{bmatrix} = \begin{bmatrix} -\frac{\partial W(P_i)_1}{\partial M_1} & \cdots & -\frac{\partial W(P_i)_1}{\partial M_m} \\ \vdots & \ddots & \vdots \\ -\frac{\partial W(P_i)_d}{\partial M_1} & \cdots & -\frac{\partial W(P_i)_d}{\partial M_m} \end{bmatrix}$$



Levenberg-Marquardt

n data points

$$E(M) = \sum_{i=1}^n \|r_i(M)\|^2, \text{ where residual } r_i(M) = O_i - W(P_i): O \text{ is } d\text{-dimensional, } M \text{ is } m\text{-dimensional}$$

Solve for change to M at each step: $(J^T J + \lambda \text{diag}(J^T J)) \Delta M = -J^T r$ (Linear system)

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Jacobian: Derivative of every residual wrt M at every data point

$r = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$

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r is implicit:
simplify notation

Jacobian $J(r) = \begin{bmatrix} J_1(r) \\ \vdots \\ J_n(n) \end{bmatrix}_{n \times d \times m}$, where $J_i(r) = \begin{bmatrix} \frac{\partial r_{i1}}{\partial M_1} & \cdots & \frac{\partial r_{i1}}{\partial M_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_{id}}{\partial M_1} & \cdots & \frac{\partial r_{id}}{\partial M_m} \end{bmatrix} = \begin{bmatrix} -\frac{\partial W(P_i)_1}{\partial M_1} & \cdots & -\frac{\partial W(P_i)_1}{\partial M_m} \\ \vdots & \ddots & \vdots \\ -\frac{\partial W(P_i)_d}{\partial M_1} & \cdots & -\frac{\partial W(P_i)_d}{\partial M_m} \end{bmatrix}$

Levenberg-Marquardt Example

Consider
'Homography'
(scale independent)

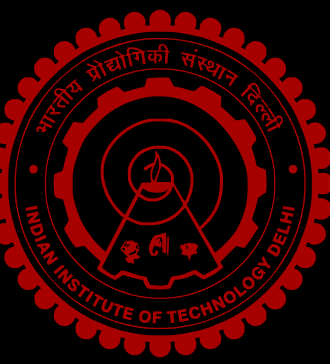
$$\begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} = \begin{bmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} \quad M_1 \text{ to } M_9$$

$$\begin{bmatrix} r_{i1}(M) \\ \vdots \\ r_{id}(M) \end{bmatrix} = \begin{bmatrix} m_{00}x_i + m_{01}y_i + m_{02} - u_i \\ m_{10}x_i + m_{11}y_i + m_{12} - v_i \\ m_{20}x_i + m_{21}y_i + m_{22} - w_i \end{bmatrix}$$

$$\Rightarrow J_i = \begin{bmatrix} x_i & y_i & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_i & y_i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_i & y_i & 1 \end{bmatrix}$$

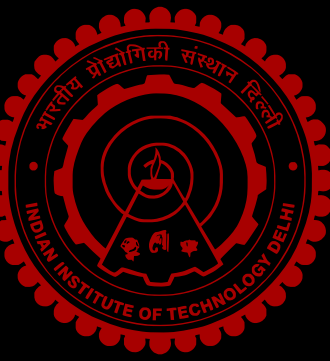
$$r_i = \begin{bmatrix} r_{i1}(M) \\ \vdots \\ r_{id}(M) \end{bmatrix}$$

Jacobian $J = \begin{bmatrix} J_1 \\ \vdots \\ J_n \end{bmatrix}_{n \times d \times m}$, where $J_i = \begin{bmatrix} \frac{\partial r_{i1}}{\partial M_1} & \cdots & \frac{\partial r_{i1}}{\partial M_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_{id}}{\partial M_1} & \cdots & \frac{\partial r_{id}}{\partial M_m} \end{bmatrix}$



Levenberg-Marquardt Iteration

- Start with initial guess for M and λ and iterate to improve:
 1. Evaluate residuals $r_i(M)$, and Jacobian J
 2. Solve the linear LM system for the step ΔM : $(J^\top J + \lambda \text{diag}(J^\top J))\Delta M = -J^\top r$
 3. Try ΔM : $M_{\text{try}} = M + \Delta M$, Compute residuals $r(M_{\text{try}})$
 - Compute Gain Ratio $\rho = [E(M_{\text{try}}) - E(M)] / \frac{1}{2}\Delta M^\top [\lambda \text{diag}(J^\top J)\Delta M - J^\top r]$
 - $\rho > 0.25 \Rightarrow$ accept trial, update $M = M_{\text{try}}$, If $\|\Delta M\| < \Delta_{\text{ok}}$, M may be an acceptable solution
 - If $\rho > 0.75$, reduce λ for more aggressive step
 - $\rho \leq 0.25 \Rightarrow$ reject trial: Discard M_{try} , Increase λ to slow descent, continue to step 2



Handling Noisy Data

RANSAC-based

- Start with several randomly selected small subsets of data
 - RANSAC chooses each subset as the *minimal* points required for model fitting
- Compute Model parameters for each subset and test it on full samples
 - Outliers are points P_i , for which $r_i > \tau$ for some chosen τ , others are inliers
- The model with the most inliers wins
 - Optionally: Also consider $E(\text{inliers})$ among the best few models and choose the smallest
- Finally refine the top model by re-fitting with all its inliers