MFCS: Problem set 3

1 s-m-n theorem

1. Show that there is a total computable function k such that for each n, $k(n)$ is an index of the function $[\sqrt[n]{x}]$.
2. Show that there is a total computable function k such that for each n, $W_{k(n)} =$ the set of perfect n^{th} powers.
3. Let $n \geq 1$. Show that there is a total computable function s such that

$$W_{s(x)}^{(n)} = \{(y_1, \ldots, y_n) : y_1 + y_2 + \ldots + y_n = x\}$$

4. Show that the functions s_n^m defined in the proof of the $s-m-n$ theorem are primitive recursive.
5. Show that for each m there is a total $(m+1)$-ary computable function s^m such that for all n

$$\phi_e^{(m+n)}(\mathbf{x}, \mathbf{y}) \simeq \phi_{s^m(e,\mathbf{x})}(\mathbf{y})$$

   where $\mathbf{x}$ and $\mathbf{y}$ are m- and n-tuples respectively.

2 Universal programs and universal functions

1. (a) Show that there is a decidable predicate $Q(x, y, z)$ such that
      i. $y \in E_x$ if and only iff $\exists z Q(x, y, z)$
      ii. if $y \in E_x$, and $Q(x, y, z)$, then $\phi_x((z)_1) = y$
   (b) Deduce that there is a computable function $g(x, y)$ such that
      i. $g(x, y)$ is defined if and only if $y \in E_x$
      ii. if $y \in E_x$, then $g(x, y) \in W_x$ and $\phi_x(g(x, y)) = y$; i.e. $g(x, y) \in \phi_x^{-1}(\{y\})$

(c) Deduce that if f is a computable injective function (not necessarily total or surjective) the f^{-1} is computable.

2. Show that there is total computable function $k(e)$ such that for any e, if ϕ_e is the characteristic function for a decidable predicate $M(x)$, then $\phi_{k(e)}$ is the characteristic function for $\neg M(x)$.

3. Show that there is a total computable function $k(x)$ such that for every x, $E_{k(x)} = W_x$.

4. Show that there is a total computable function $s(x, y)$ such that for all x, y $E_{s(x,y)} = E_x \cup E_y$.

5. Suppose that $f(x)$ is computable; show that there is a total computable function $k(x)$ such that for all x, $W_{k(x)} = f^{-1}(W_x)$.

3 Decidability and Partial decidability

1. Solve the problems of Assignment-2 using Rice's theorem.

2. Show that the following predicates are partially decidable:

   (a) '$E_x^{(n)} \neq \phi$ (n fixed).

   (b) '$\phi_x(y)$ is a perfect square'.

   (c) 'n is a Fermat's number'. (we say the n is a Fermat's number if there are numbers $x, y, z > 0$ such that $x^n + y^n = z^n$.).

   (d) 'there is a run of x consecutive 7's in the decimal expansion of π'.

3. Suppose that $M(\mathbf{x})$ and $N(\mathbf{x})$ are partially decidable. Prove that '$M(\mathbf{x})$ *and* $N(\mathbf{x})$' and '$M(\mathbf{x})$ *or* $N(\mathbf{x})$' are partially decidable. Show that '$\neg M(\mathbf{x})$' is not necessarily partially decidable.

4. Suppose that $M(\mathbf{x}, y)$ is partially decidable. Show that

   (a) '$\exists y < z M(\mathbf{x}, y)$' is partially decidable.

   (b) '$\forall y < z M(\mathbf{x}, y)$' is partially decidable.

   (c) '$\forall y M(\mathbf{x}, y)$' is not necessarily partially decidable.

5. Show that the following predicated are diophontine:

   (a) 'x is even'

   (b) 'x divides y'

6. Technique of reducibilty to show that a predicate is not partially decidable.

   (a) Suppose that $M(x)$ is a predicate and k a total computable function such that $x \in W_x$ iff $M(k(x))$ does not hold. Prove that $M(x)$ is not partially decidable.

   (b) Prove that 'ϕ_x is total' is not partially decidable.

   (c) By considering the function

   $$f(x,y) = \begin{cases} 1 & \text{if } P_x(x) \text{ does not converge in } y \text{ or fewer steps} \\ undefined & \text{otherwise} \end{cases}$$

   show that 'ϕ_x is total' is not partially decidable. [Hint: use s-m-n theorem and (a)].

7. Suppose that $M(x_1, \ldots, x_n)$ is partially decidable and $g_1, \ldots, g_n$ are computable partial functions. Show that $N(\mathbf{y})$ given by

   $$N(\mathbf{y}) = M(g_1(\mathbf{y}), \ldots, g_n(\mathbf{y}))$$

   is partially decidable. (we take this to mean that $N(\mathbf{y})$ does not hold if any one of $g_1(\mathbf{y}), \ldots, g_n(\mathbf{y})$ is undefined.)

4 Recursive and recursively enumerable sets

1. Let A, B be subsets of $\mathbb{N}$. Define $A \oplus B$ and $A \otimes B$ by

   $$A \oplus B = \{2x : x \in A\} \cup \{2x+1 : x \in B\}$$
   $$A \otimes B = \{\pi(x,y) : x \in A \text{ and } y \in B\},$$

   where $\pi(x,y) = 2^x(2y+1) - 1$ defined before. Prove that

   (a) $A \oplus B$ is recursive iff both A and B are recursive.

   (b) If $A, B \neq \phi$, then $A \otimes B$ is recursive iff both A and B are recursive.

2. Let $B \subseteq \mathbb{N}$ and let $n > 1$. Prove that if B is recursive then the predicate given by $M(x_1, \ldots, x_n)$ given by

   $$M(x_1, \ldots, x_n) = \text{‘}2^{x_1}3^{x_2}\ldots p_n^{x_n} \in B\text{’}$$

   is decidable.

3. Let $A \in \mathbb{N}^n$. Prove that A is recursive iff $\{2^{x_1}3^{x_2}\dots p_n^{x_n} : (x_1, \dots, x_n) \in A)\}$ is recursive.

4. For any $a \in \mathbb{N}$, let ${}^aW_x = \{x : \phi_e(x) = a\}$. Show that ${}^aW_x$ is r.e for all a. Does the enumeration ${}^aW_0, {}^aW_1, {}^aW_2, \dots$ include all r.e. sets?

5. Show that the set $\{x : \phi_x \text{ is not injective}\}$ is r.e.

6. Show that there are total computable functions k, l such that for every x, $w_x = E_{k(x)}$ and $E_x = W_{l(x)}$.

7. Suppose that A is an r.e set. Show that the sets $\cup_{x\in A} W_x$ and $\cup_{x\in A} E_x$ are both r.e. Show that $\cap_{x\in A} W_x$ is not necessarily r.e. as follows. For ant t let $K_t = \{x : P_x(x) \downarrow \text{ in } t \text{ steps}\}$. Show that for ant t, K_t is recursive; moreover $K = \cup_{t\in\mathbb{N}} K_t$ and $\bar{K} = \cap_{t\in\mathbb{N}} \bar{K}_t$.

8. Let f be a unary computable function, and suppose that $A \subseteq Dom(f)$, and let $g = f \mid A$. Prove that g is computable iff A is r.e.

9. Let f is a unary function. Prove that f is computable iff the set $\{2^x 3^{f(x)} : x \in Dom(f)\}$ is r.e.

10. Let A be an inifinite r.e. set. Show that A can be enumerated without repetitions by a total computable function.

11. Which of the following sets are recursive? Which are r.e.? Which have r.e. complement?

    (a) $\{x : x \in E_x\}$

    (b) $\{x : x \text{ is a perfect square }\}$

    (c) $\{x : \phi_x \text{ is injective}\}$

    (d) $\{x : \text{ there is run of at least } x \text{ consecutive 7s in the decimal expansion of } \pi\}$

    (e) $\{x : P_m(x) \uparrow\}$ (m is fixed)

12. (a) Let $B \subseteq \mathbb{N}$ and let $n > 1$. Prove that id B is r.e then the predicate $M(x_1, \dots, x_n)$ given by

    $$M(x_1, \dots, x_n) = \text{‘}2^{x_1}3^{x_2}\dots p_n^{x_n} \in B\text{’}$$

    is partially decidable.

    (b) Let $A \in \mathbb{N}^n$. Prove that A is r.e. iff $\{2^{x_1}3^{x_2}\dots p_n^{x_n} : (x_1, \dots, x_n) \in A)\}$ is r.e..

(c) Prove that $A \in \mathbb{N}^n$ is r.e. iff $A = \phi$ or there is a total computable function $\mathbf{f} : \mathbb{N} \to \mathbb{N}^n$ such that $A = Ran(f)$. (By a total computable function $\mathbf{f}$ from $\mathbb{N}$ to $\mathbb{N}^n$ we mean an n-tuple $\mathbf{f} = (f_1, \ldots, f_n)$ where each f_i is a unary computable function and $\mathbf{f}(x) = (f_1(x), \ldots, f_n(x))$.)

13. Suppose that f is a total computable function, A is a recursive set and B is a recursively enumerable set. Show that $f^{-1}(A)$ is recursive and that $f(A)$, $f(B)$ and $f^{-1}(B)$ are r.e. but not necessarily recursive. What extra information can be obtained about these sets if f is a bijection?

14. Use Rice-Shapiro's theorem to show that the following problems are not partially decidable:

    (a) '$W_x = \phi$'

    (b) 'W_x is finite'

    (c) 'W_x is infinite'

    (d) '$\phi_x = \mathbf{0}$'

    (e) '$\phi_x \neq \mathbf{0}$'

15. Prove Rice's theorem from the Rice-Shapiro's theorem. [Hint: Suppose that '$\phi_x \in \mathcal{B}$' is decidable. Then both $\mathcal{B}$ and $\mathcal{C}_1 \backslash \mathcal{B}$ satisfy the conditions of Rice-Shapiro: consider the cases $f_\phi \in \mathcal{B}$ and $f_\phi \notin \mathcal{B}$.].