MFCS: Problem set 1

1 URM programs

1. Show that the following functions are computable by devising programs that compute them.

   (a)

   $$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases}$$

   (b)

   $$f(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

   (c)

   $$f(x, y) = \begin{cases} 0 & \text{if } x \leq y \\ 1 & \text{if } x > y \end{cases}$$

   (d)

   $$f(x) = \begin{cases} x/3 & \text{if } x \text{ is a multiple of 3} \\ \text{undefined} & \text{otherwise} \end{cases}$$

   (e) Suppose that P is a program without jump instructions. Show that there is a number m such that either $f_P^{(1)}(x) = m \forall x$ or $f_P^{(1)}(x) = x + m \forall x$

(f) Show that for each transfer instruction $T(m,n)$ there is a program without transfer instructions that has exactly the same effects as $T(m,n)$ on any configuration of the URM. (Thus transfer instructions are really redundant in the formulation of URM; it is nevertheless convenient to have transfer as a basic facility of the URM.)

2. Show that the following predicates are decidable:

   (a) '$x < y$'

   (b) '$x \neq 3$'

   (c) 'x is even'

2 Substitution, primitive recursion and minimalization

Do the following without writing any programs.

1. Show that for every $m \in \mathbb{N}$ the following functions are computable:

   (a) $\mathbf{m}$ (recall that $\mathbf{m}(x) = m \forall x$)

   (b) mx

2. Suppose that $f(x,y)$ is a computable function and $m \in \mathbb{N}$. Show that the function $h(x) \simeq f(x,m)$ is computable.

3. Suppose that the function $g(x)$ is a total computable function. Show that the predicate $M(x,y) \equiv$ '$g(x) = y)'$ is decidable.

4. Show that the following functions are computable:

   (a) Any polynomial function $a_0 + a_1 x + \ldots + a_n x^n$ where $a_0, a_1, \ldots a_n \in \mathbb{N}$

   (b) $[\sqrt{x}]$

   (c) $LCM(x,y)$

   (d) $HCF(x,y)$

   (e) the number of prime divisors of x

   (f) the number of positive integers less that x which are relatively prime to x. (x and y are relatively prime if $HCF(x,y) = 1$.)

(g) $f(0) = 1; f(1) = 2; f(n+2) = f(n) + f(n+1)$. ($f(n)$ is the n^{th} Fibonacci number.)
(**Hint:** show that the function $2^{f(x)}3^{f(x+1)}$ is computable using recursion.)

5. Show that the following are decidable:
   (a) x is odd
   (b) x is a power of a prime number
   (c) x is a perfect cube

6. Suppose that $f(x)$ is a total injective computable function. Prove that f^{-1} is computable.

7. Suppose that $p(x)$ is a polynomial with integer coefficients. Show that the function $f(a)$ = least non-negative integral root of $p(x) - a$ where $a \in \mathbb{N}$ is computable ($f(a)$ is undefined if there is no such root.)

8. Show that the function

$$f(x, y) = \begin{cases} x/y & \text{if } y \neq 0 \text{ and } y \mid x \\ \text{undefined} & \text{otherwise} \end{cases}$$

is computable.

3 Numbering computable functions

1. NOTE: MISTAKE IN WHAT WAS DONE IN THE CLASS

   the bijection $\tau : \cup_{k>0}\mathbb{N}^k \to \mathbb{N}$ should be defined as

   $\tau(a_1, \dots, a_n) = 2^{a_1} + 2^{a_1+a_2+1} + 2^{a_1+a_2+a_3+2} + \dots 2^{a_1+a_2+\dots+a_k+k-1}$

   Only then the bijection given in the class holds. Work out the complete proof that τ is an effective bijection. How come nobody pointed out the mistake in the class?

2. Find
   (a) $\beta(J(3, 4, 2))$
   (b) $\beta^{-1}(503)$

(c) the code of the following program:

$$T(3,4), S(3), Z(1)$$

(d) P_{100}

3. Prove that every computable function has infinitely many indices.

4. Suppose that $f(x,y)$ is a total computable function. For each m, let g_m be the computable function given by $g_m = f(m,y)$. Construct a total computable function h such that for each m, $h \neq g_m$.

5. Let $f_0, f_1, \ldots$ be an enumeration of partial functions from $\mathbb{N}$ to $\mathbb{N}$. Construct a function g from $\mathbb{N}$ to $\mathbb{N}$ such that $Dom(g) \neq Dom(f_i)$ for each i.

6. Let f be a partial function from $\mathbb{N}$ to $\mathbb{N}$, and let $m \in \mathbb{N}$. Construct a non-computable function g such that $g(x) \simeq f(x)$ for $x \leq m$.

7. Show that the set of all functions from $\mathbb{N}$ to $\mathbb{N}$ is not denumerable.