CS 909, (Randomized Algorithms) Assignment Sheet 2 (Due Mar 6, 03)

1. Design a 1-1 permutation routing for an $N \times N$ grid that takes $O(N)$ rounds with high probability. Analyse the following strategy. For a packet originating at i, j with destination i', j', first route to a random random row within column j, say r_i, j. In the second phase route to r_i, j' (destination column). In the third phase route to i', j' (destination row).
(i) Analyze this algorithm assuming that an edge can simultaneously route two packets - one in each direction.
(ii) Show how to route in $2n + o(n)$ steps whp which is optimal upto the leading constant.

2. **Lower bound for oblivious routing**. Let $G = (V, E)$ be a network with $|V| = N$ nodes and maximum degree d. We would like to show that there are instances of $1 - 1$ permutation routing for which any oblivious algorithm will take $\Omega(n)$ rounds where $n = \sqrt{N}/d$.
Define $P_{u,v}$ be the (oblivious) path from node u to node v. We want to construct an instance $P_{u_1,v_1}, P_{u_2,v_2} \ldots P_{u_k,v_k}$ that use a common edge (and hence congestion is at least k). For a fixed node v, consider the $N - 1$ paths $P_{u,v}, u \in \{V - v\}$ ending at v. Let $S_k(v)$ be the set of edges that have congestion k among $P_{u,v}$ and $S_k^*(v)$ be the set of vertices incident on $S_k(v)$.
(i) For $k \leq \frac{N-1}{d}$, show that
$$|V - S_k^*(v)| \leq (k-1)(d-1)|S_k^*(v)|$$
$$\text{(ii)} \quad |S_k(v)| \geq \frac{N}{2kd}$$
(iii) For $k = n = \sqrt{N}/d$, show that there is some edge e that belongs to at least n different $S_k(v)$ (for different v's).
(iv) Construct a routing instance such that n paths pass through e.
(v)** There are at least $(\sqrt{N}/d)!$ routing instances that require $\sqrt{N}/d$ rounds.

3. Show that the expected number of rotations required to insert an element into a treap is 2.
Hint: Pretend that the priority of the inserted element is minimum when it is inserted and increase it incrementally till it achieves the required priority. As you increase the priority larger than the ith priority, when would you require a rotation ?

4. Show that the expected length of the (unique) random path between two nodes x_1 and x_2 of a treap is $\log r$ where r is the difference of the priorities of the two nodes (in the range $1..n$).

5. Describe an $O(n)$ algorithm to generate a random permutation. Prove rigorously that all permutations are equally likely.

6. Given a set of points on the real-line with coordinates $x_1, x_2 \ldots x_n$, we want to determine if there is a subset $x_i, x_{i+1} \ldots x_{i+m-1}$ with separation distances d_i $i \leq m$. Design a $O(n)$ algorithm for this problem.