COL 702 Lecture 12 7 Sept

Given a subset system $M = (S, \mathcal{I})$

ground set

Independent subsets or feasible subsets

Recall that if $S \in \mathcal{I}$ then $S' \subset S$ $S' \in \mathcal{I}$

The following are equivalent

1. M is a matroid, i.e. the generic greedy algorithm solves the optimization problem for any weight function $w: S \to R^+$

2. for any $S, S' \in \mathcal{I}$ and $|S'| < |S|$ $\exists$ $e \in S - S'$ s.t. $S' \cup \{e\} \in \mathcal{I}$

(Exchange property)

3. For any $A \subset S$, the all the "maximal" subsets of A have the same cardinality

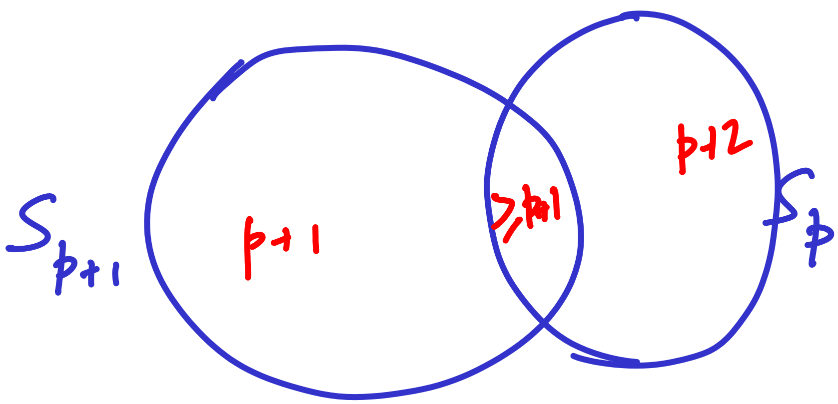
(rank property)

Show that ① $\Rightarrow$ ② $\Rightarrow$ ③ $\Rightarrow$ ①

Suppose the given subset system is a Matroid (greedy works for all weight fns)

Suppose property 2 doesn't hold.

It implies that there are subsets S_{p+1}, S_p s.t. $|S_{p+1}| = p+1$ $|S_p| = p$ and for no element $e \in S_{p+1} - S_p$ $\{e\} \cup S_p$ is independent



Consider the following weight function for the elements in S

$$w(e) = \begin{cases} p+2 & \text{if } e \in S_p \\ p+1 & \text{if } e \in S_{p+1} - S_p \\ 0 & \text{otherwise} \end{cases}$$

What would greedy do

It will choose all elements from S_p (and can't add any more wt) greedy soln : $(p+2)p$

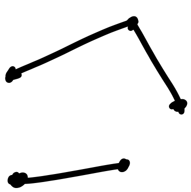
Is there a better soln?
Elements of S_{p+1} have weight $\geq (p+1)^2$

Application to MST (maximal)

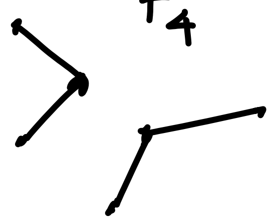
$M = (S, \mathcal{I})$ S: E set of edges
$G=(V,E,w)$ $\mathcal{I}$: all possible forests including ϕ

We want to prove that for any two forests F_{p+1} F_p we can add an edge $e \in F_{p+1} - F_p$ to F_p s.t. $\{e\} \cup F_p$ is a forest

F_3



F_4



Notation
$V(F)$. set of vertices on which edges of F are induced

$$|V(F_4)| \overset{?}{>} |V(F_3)|$$

or $V(F_3) \subset V(F_4)$

If there exists a vertex $w \in V(F_{p+1}) - V(F_p)$

then we can add any edge from F_{p+1} that is incident on w to F_p without inducing a cycle

(Observation)

If $V(F_{p+1}) \subset V(F_p)$ then

the # connected components in F_p must be more that # CC in F_{p+1}

CC = # vertices − # edges

$$V(F_p) > V(F_{p+1})$$

$$|F_{p+1}| > |F_p|$$

$$\# CC_p = V(F_p) - p$$

$$\# CC_{p+1} = V(F_{p+1}) - (p+1)$$

$$\# CC_p > \# CC_{p+1}$$

Proof : Colour the CC of F_p using distinct colours so that vertices in the same component gets the same colour

In CC of F_{p+1}, we will find an edge (u,v) connecting vertices of different colour. So u, v can be added to F_p w/o creating any cycle

(Greedy works correctly for Maximal S.T.)

<u>How about minimal Sp Tree ?</u>

$$w'(e) = W - w(e)$$

where $W = \max_e w(e)$

We are running the greedy algorithm on the increasing sequence of weights $w(e)$

<u>Claim</u>. It gives the minimal sp tree

Suppose we ran the greedy algorithm on $w'(e)$ and found the Maximal Sp Tree

From our previous proof of matroids we should get the maximum sp tree on the weights w'

Suppose this chooses the edges

$e_1 \quad e_2 \quad e_3 \quad \cdots \quad e_k$

$w'(e_1) \geq w'(e_2) \geq w'(e_3) \quad \cdots \quad \geq w(e_k')$

$$\sum_{i=1}^{k} w'(e_i) = \sum_{i=1}^{k} W - w(e_i) = kW - \sum_{i=1}^{k} w(e_i)$$

$\underline{k}$ is the size of the "maximal" independent set and is dependent only on the given graph

kW : a fixed quantity for a given graph and wt function

So $\sum_{i=1}^{k} w(e_i)$ must be minimized to maximize $kW - \sum w(e_i)$

minimum sp tree