COL 702 Lecture 24 Oct 29

Strong connectivity in directed graphs

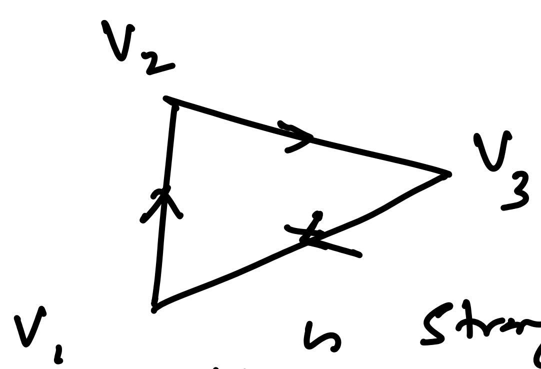
Two vertices $u, w \in V$ are said to be strongly connected if $u \underset{G}{\rightsquigarrow} w$ and $w \underset{G}{\rightsquigarrow} u$

__Observation__: This is an equivalence reln

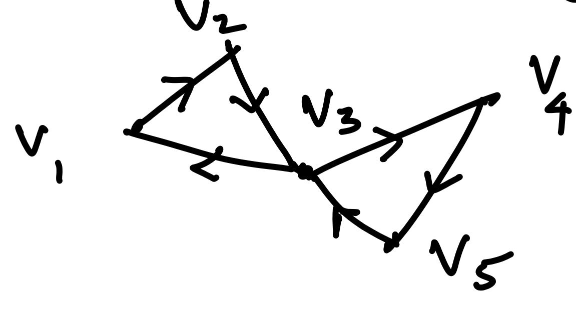
(i) reflexive (ii) symmetric (iii) $u \rightsquigarrow w$, $w \rightsquigarrow v$ $\Rightarrow u \rightsquigarrow v$ transitive

The maximal equivalence classes of this reln are called Strong Connected Components (SCC)

If the graph has one SCC then graph is strongly connected

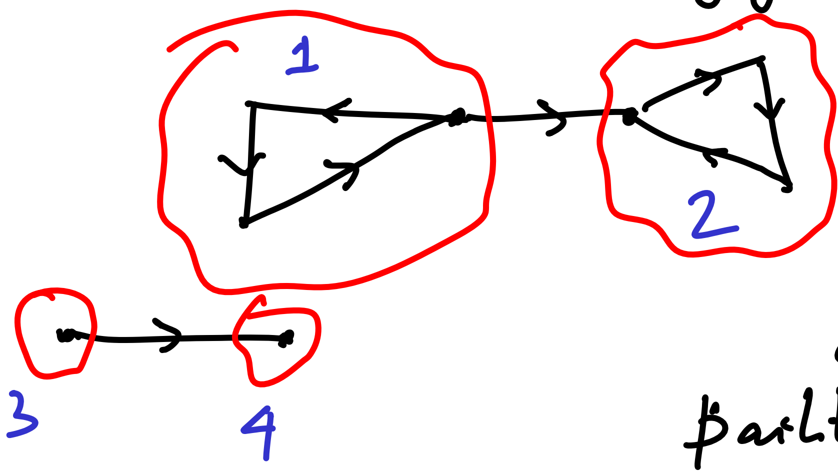


is strongly connected



is not S.C.

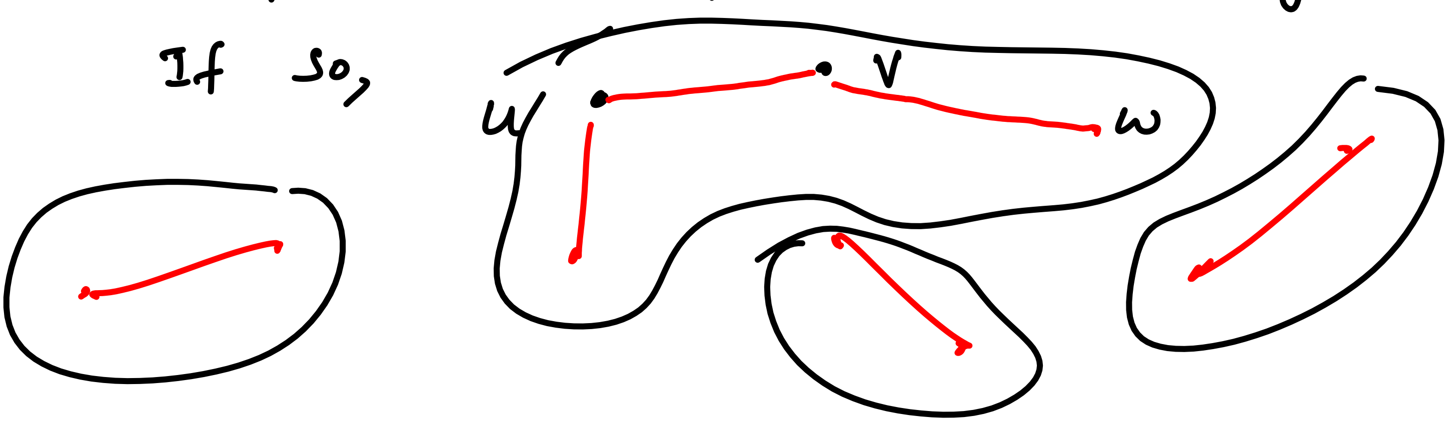
Prob : Given a directed graph, find all strongly connected components



Any equ reln induces partition called equivalence classes and S.C.C is partition of vertices

For every pair of vertices u, w find out if u, w are strong connected

If so,

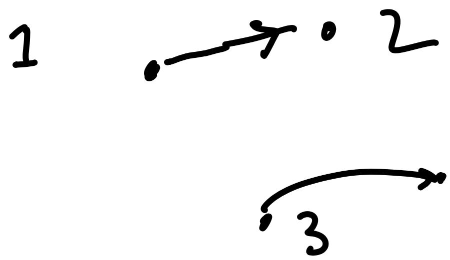


Reachability tree from every vertex will yield the requisite connectivity information

$\Rightarrow$ $O(|V| \cdot |E|)$

Can we obtain a linear time algorithm

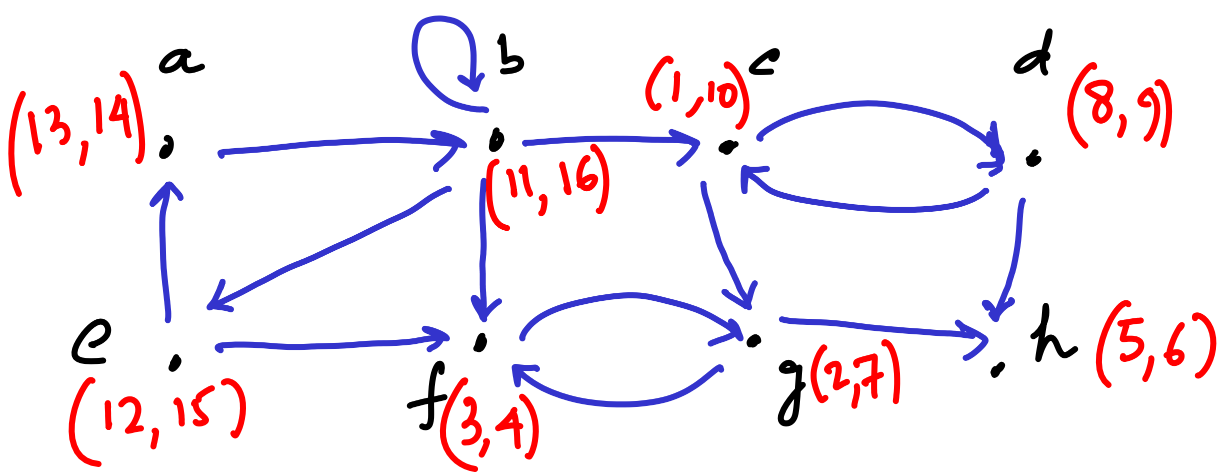
Component Graph

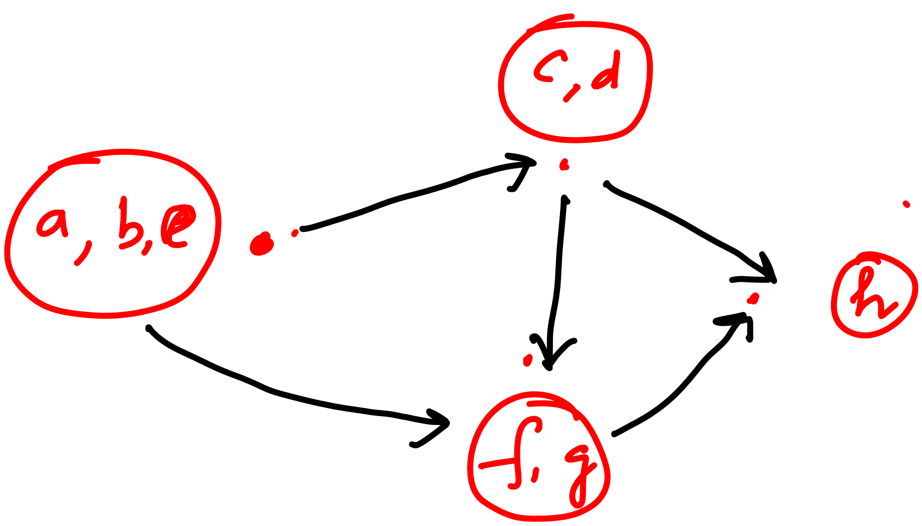


Claim: G is a DAG

(If there is a cycle then it violates maximality)

Since it is a DAG, there will be source and sink vertices (components in original graphs)





__Claim__ : The vertex v' having the largest finish time is in a source component

Then all vertices that can reach v' belong to the source component

We actually do a DFS on the reverse of the initial graph G^R

the SCC of G^R and G are identical

The component graph $\mathcal{G}^R$ and $\mathcal{G}$ will have the edges reversed. So the sources and sinks will be flipped.

Start with vertex v' with largest finish time in G^R

Then all the vertices reachable from v' in G will be the first source component of $\mathcal{G}^R$ (sink component in $\mathcal{G}$)

Repeat till all components are output

$$\boxed{\text{Two DFS (one in } G \text{ and } G^R) \text{ suffices to find SCC}}$$

Graph Spanner

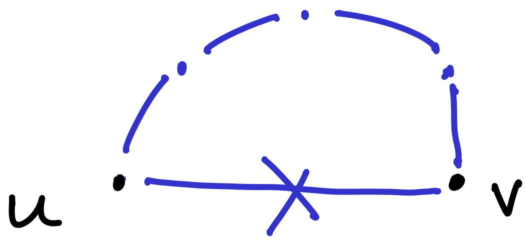
Given a weighted undirected graph, $G = (V, E, w)$ we want to construct a subgraph $G_s = (V, E_s, w)$ where $E_s \subset E$ s.t.

$$\delta_{G_s}(u, v) \leq \alpha \cdot \delta_G(u, v)$$

δ: shortest path distance $\quad$ α: is some constant ≥ 1

There is clearly a trade-off between α and $|E_s|$

If $E = E_s$ then $\alpha = 1$



$E_s \subset E$

Length of the cycle is g

then the minimum increase in (ratio) path length is $g-1$ even in an unweighted graph

This implies that the α is at least proportional to the smallest cycle in the original graph.

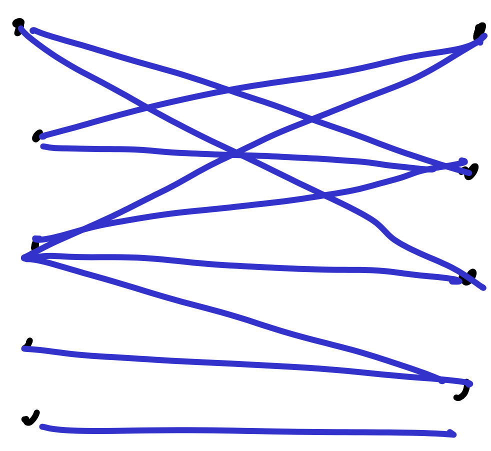
smallest cycle : girth of graph

<u>Ex</u>

Consider a complete bipartite graph

n vertices

n vertic



n^2 edges

Throw away one edge

$\Rightarrow \alpha' \geq 3$