Shortest paths

Given a directed graph $G = (V, E)$ with weights $\omega : E \to \mathbb{R}$

(i) For $s, t \in V$ find the shortest path from s to t

(ii) Given $s \in V$ find the shortest path from s to all $v \in V$

Single Source Shortest Path : Dijkstra, Bellman-Ford

(iii) All pair shortest paths from s to t APSP

Floyd-Warshall

A path P is a sequence of edges $e_1 = (u_1, v_1)$ $e_2 = (u_2, v_2)$ $e_i = (u_i, v_i)$ such that $v_i = u_{i+1}$ and $u_1 = s$, $v_k = t$

The weight $\omega(P) = \sum_i \omega(e_i)$

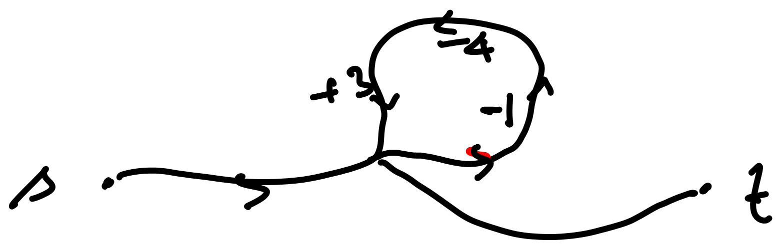
Length(P) = # edges/# hops

Nature of the weight function

ω : Dijkstra non-negative "Easier to solve"

: negative → BF no negative cycles

→ negative cycles



Only simple paths are allowed (no vertices repeated)

lengths $\leq |V|-1$

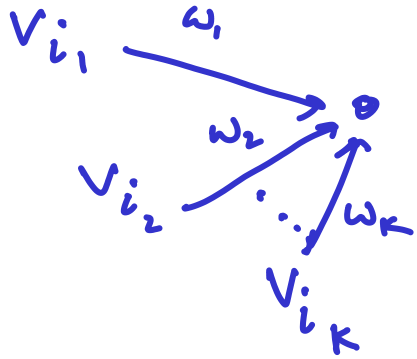
All pair shortest path (APSP)

Some properties of shortest paths

Subpath optimality

If $v_0 \; v_1 \; v_2 \; \ldots \; v_k$ is a shortest path between $v_0 = s$ and $v_k = t$

then $v_i \; v_{i+1} \; \ldots \; v_\ell$ is also a shortest path between v_i and v_ℓ

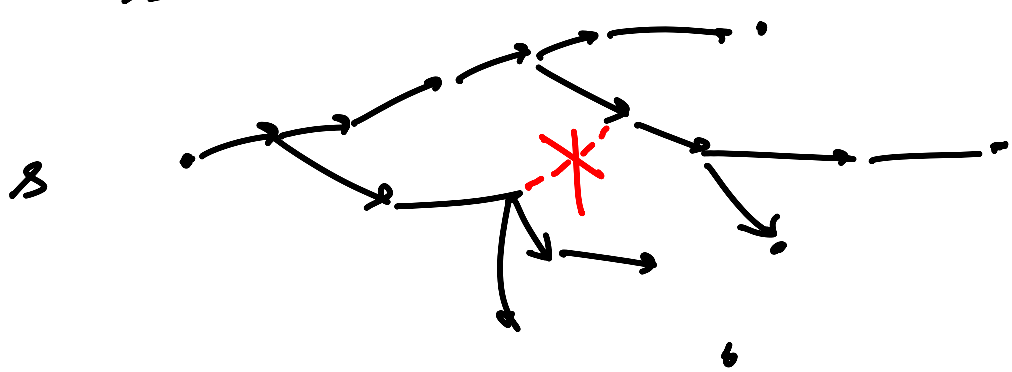


The shortest path distance $\delta(v)$: shortest path distance from s to v

$$\delta(v) = \min_{v_{i_1}\dots v_{i_K}} \left\{ \delta(v_{i_j}) + w_j \right\}$$

set of all neighbors of v s.t. $(v_{i_1}, v) \in E$

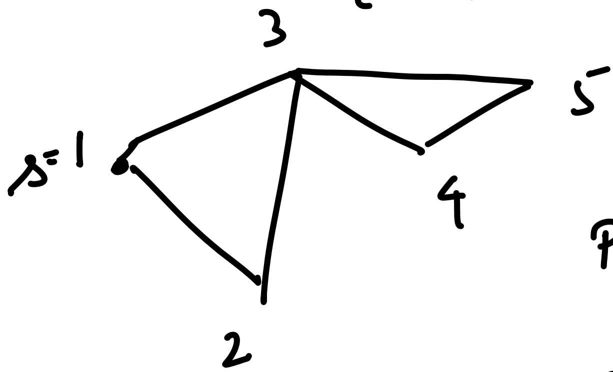
The actual shortest path (not distances) actually form a directed-tree rooted at s



Induction for Floyd Warshall

P^i : is a path from s to v that does not use any vertex numbered greater than i, not including s, v

$V = \{1, 2, \ldots n\}$



$P_1 = [1, 3, 4]$ $\qquad$ $P_2 = [1, 2, 3]$

$P_1 \cancel{\in \mathcal{P}^4}$ $\qquad$ $P_2 \cancel{\in} \mathcal{P}^3$

$P_1 \in \mathcal{P}^3$ $\qquad$ $P_2 \in \mathcal{P}^2$

Floyd Warshall approach is to

begin with $\mathcal{P}^0$: paths consisting of single edges

Subsequently we construct $\mathcal{P}^1, \mathcal{P}^2, \ldots \mathcal{P}^n$

Suppose we have computed $\mathcal{P}^k$

$p^{(k)}(i,j)$: denotes the shortest path between vertices i, j that doesn't use vertex higher than k

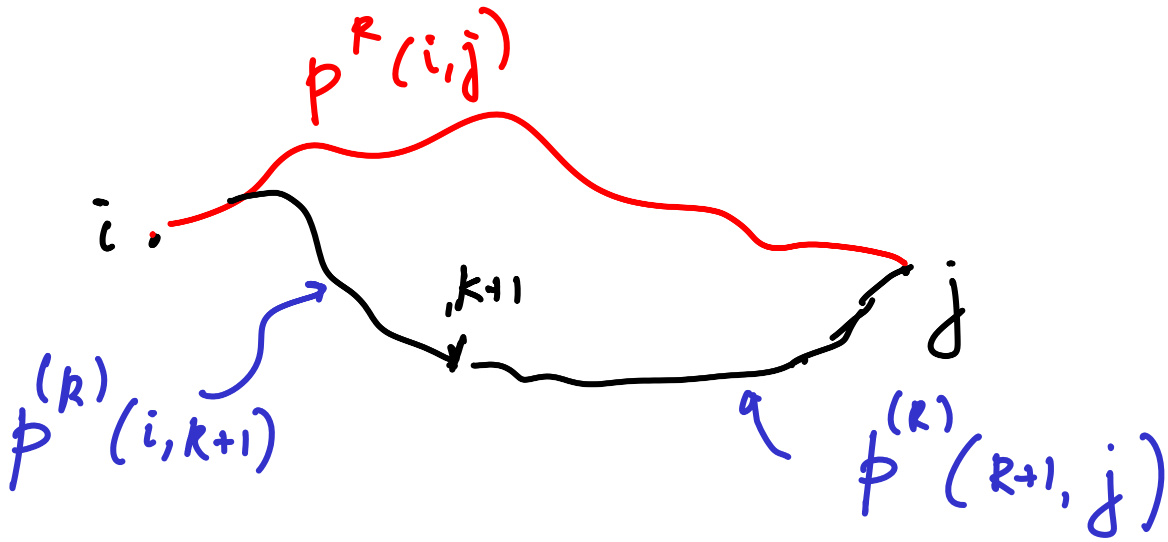
$$p^{(0)}(i,j) = \begin{cases} w(i,j) & \text{if there is an edge } (i,j) \\ \infty & \text{if there is no edge} \end{cases}$$

We have $p^{(k)}(i,j)\ \forall\ i,j \in V$

$p^{(k+1)}(i,j) =$

$\rightarrow$ $p^{(k+1)}(i,j) = p^{(k)}(i,j)$ as the shortest path doesn't go through $k+1$

$\rightarrow$



$$\min \left\{ p^{k}(i,j)\ ,\ \left[p^{(k)}(i,k+1) + p^{(k)}(k+1,j) \right] \right\}$$

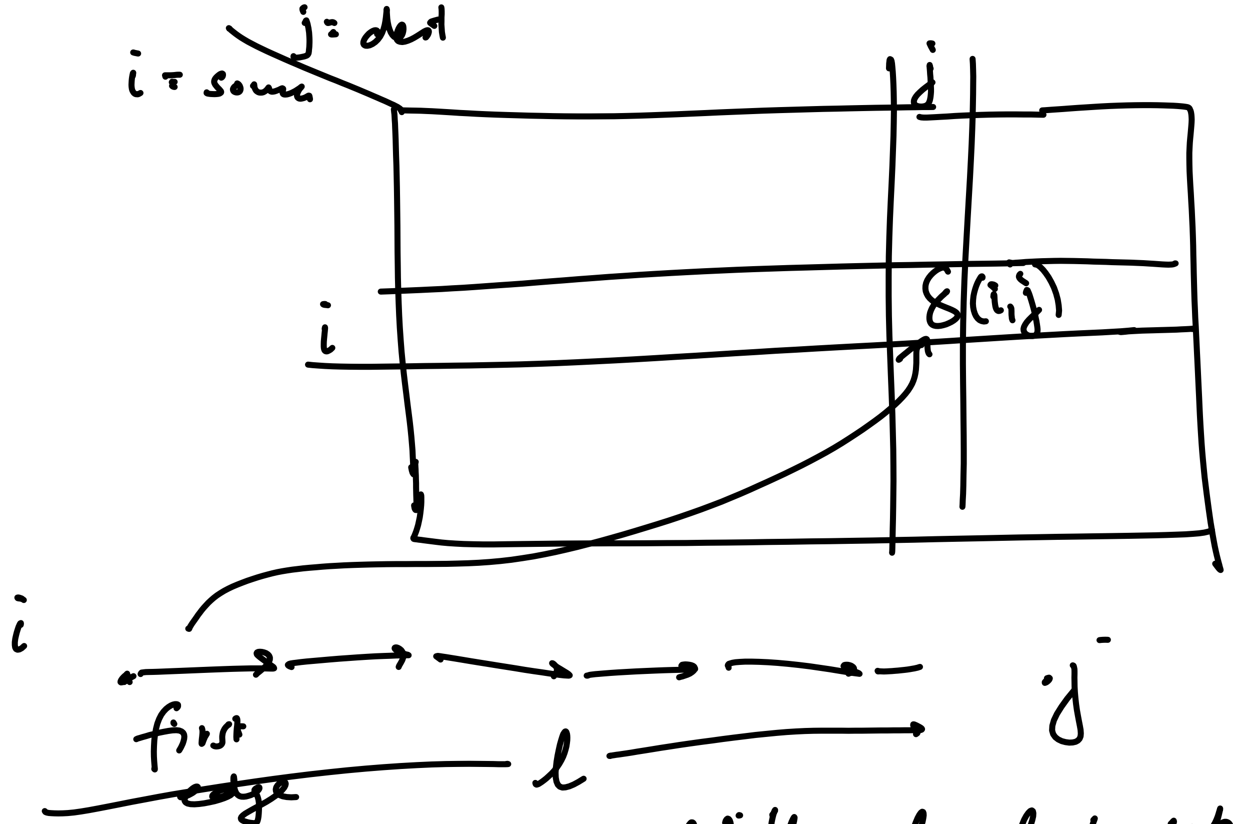
D.P. for Floyd Warshall computes n^3 entries in increasing order of k and each entry takes $O(1)$ time

$$\Rightarrow O(n^3) \text{ time}$$

$O(n^2)$ space since only Table k is needed

How to get the actual paths?

How to store all paths?



With l look ups we can reconstruct the shortest