Hennessy-Milner Logic

Neelesh Korade[1] and S. Arun-Kumar[2*]

[1] Persistent Systems Private Limited, "Bhageerath", 402, Senapati Bapat Road Pune 411016, India. email: neelesh_korade@persistent.co.in

[2] Department of Computer Science and Engineering, Indian Institute of Technology, Delhi, Hauz Khas, New Delhi 110016, India. email: sak@cse.iitd.ernet.in

0.1 Strong Simulation

Definition 1. A *binary relation* $R \subseteq \mathbb{P} \times \mathbb{P}$ is a **strong simulation (SS)** if for every $\langle p, q \rangle \in R$ and $a \in Act$:

$$p \xrightarrow{a} p' \implies \exists q' : q \xrightarrow{a} q' \wedge p' R q'$$

We write $p \lesssim q$ if there exists a strong simulation R such that pRq.

Definition 2. The class L_{SS} of *strong simulation formulas* over Act is given by the following grammar.

$$\varphi ::= \quad <a> \varphi \mid \bigwedge_{i \in \mathbf{I}} \varphi_i$$

Note that the index set I may be infinite. $tt \equiv \bigwedge_{i \in \emptyset} \varphi_i \in L_{SS}$.

Definition 3. The *satisfaction relation* $\models \subseteq \mathbb{P} \times L_{SB}$ is defined recursively by:

- $p \models tt$ for all $p \in \mathbb{P}$.
- $p \models <a> \varphi$ for $a \in Act$ if $\exists p' \in \mathbb{P} : p \xrightarrow{a} p'$ and $p' \models \varphi$.
- $p \models \bigwedge_{i \in I} \varphi_i$ if $p \models \varphi_i$ for all $i \in I$.

The set $SS(p)$ is defined as $SS(p) = \{\varphi \in L_{SS} \mid p \models \varphi\}$. We write $p \sqsubseteq_{SS} q$ iff $SS(p) \subseteq SS(q)$ and we write $p =_{SS} q$ iff $SS(p) = SS(q)$.

Theorem 4. *[3] $p \lesssim q \iff p \sqsubseteq_{SS} q$.*

Proof. ($\Longrightarrow$) Let $p \lesssim q$, and let $\varphi \in SS(p)$. By induction on the structure of φ we show $\varphi \in SS(q)$.
Since $p \lesssim q$, there exists a *strong simulation* R such that pRq.

- Let $p \models <a> \varphi$. Then there exists a $p' \in \mathbb{P}$ with $p \xrightarrow{a} p'$ and $p' \models \varphi$. Since pRq, there must be a $q' \in \mathbb{P}$ with $q \xrightarrow{a} q'$ and $p'Rq'$. So by induction $q' \models \varphi$ and hence $q \models <a> \varphi$.
- $p \models \bigwedge_{i \in I} \varphi_i \iff \forall i \in I\, (p \models \varphi_i) \overset{induction}{\iff} \forall i \in I\, (q \models \varphi_i) \iff q \models \bigwedge_{i \in I} \varphi_i$.

* corresponding author

($\Longleftarrow$) To prove that $p \sqsubseteq_{SS} q \Longrightarrow p \lesssim q$, it suffices to establish that $\sqsubseteq_{SS}$ is a simulation. Suppose, $p \sqsubseteq_{SS} q$ and $p \xrightarrow{a} p'$. We have to show that $\exists q' \in \mathbb{P}$ with $q \xrightarrow{a} q'$ and $p' \sqsubseteq_{SS} q'$. Let

$$Q = \{q'' \in \mathbb{P} \mid q \xrightarrow{a} q'' \wedge p' \not\sqsubseteq_{SS} q''\}$$

For every $q'' \in Q$ there is a formula $\varphi_{q''} \in SS(p') - SS(q'')$. Now $<a> \bigwedge_{q'' \in Q} \varphi_{q''} \in SS(p) \subseteq SS(q)$. Therefore there must be a $q' \in \mathbb{P}$ with $q \xrightarrow{a} q' \wedge q' \models \bigwedge_{q'' \in Q} \varphi_{q''}$ and which implies $q' \notin Q$. Hence $p' \sqsubseteq_{SS} q'$ and $\sqsubseteq_{SS}$ is a strong simulation. $\square$

1 Modal Characterizations of Strong and Weak Bisimilarity

Here we review Hennessy-Milner Logic (HML) and Observable Hennessy-Milner Logic (OHML) and show how they characterize strong and weak bisimilarity respectively.

Let V be a set of *visible actions*, $\tau \notin V$ a distinguished *invisible action* and $Act = V \cup \{\tau\}$ the set of *actions*. A *labelled transition system (LTS)* is a 3-tuple $\langle \mathbb{P}, Act, \longrightarrow \rangle$, where $\mathbb{P}$ is a set of *process states* or *processes* and $\longrightarrow \subseteq \mathbb{P} \times Act \times \mathbb{P}$ is the *transition relation*. We use the notation $p \xrightarrow{a} q$ to denote $(p, a, q) \in \longrightarrow$ and refer to q as a *strong a-derivative* of p.

1.1 Strong Bisimulation

Definition 5. A *binary relation* $R \subseteq \mathbb{P} \times \mathbb{P}$ is a **strong bisimulation (SB)** if both R and R^{-1} are strong simulations. We write $p \sim q$ if there exists a strong bisimulation R such that pRq.

In [1] the author gives a comprehensive account of the modal logic (HML) that characterizes strong bisimulation. It is defined as follows.

Definition 6. The class L_{SB} of *strong bisimulation formulas* over Act is given by the following grammar.

$$\varphi ::= \quad <a> \varphi \mid \bigwedge_{i \in \mathbf{I}} \varphi_i \mid \neg\varphi$$

$tt \in L_{SB}$ where $tt \equiv \bigwedge_{i \in \emptyset} \varphi_i$. Similarly, $ff \in L_{SB}$ where $ff \equiv \neg tt$.

Definition 7. The *satisfaction relation* $\models \subseteq \mathbb{P} \times L_{SB}$ is defined recursively as follows:

- $p \models tt$ for all $p \in \mathbb{P}$.
- $p \models <a> \varphi$ for $a \in Act$ if $\exists p' \in \mathbb{P} : p \xrightarrow{a} p'$ and $p' \models \varphi$.
- $p \models \bigwedge_{i \in I} \varphi_i$ if $p \models \varphi_i$ for all $i \in I$.
- $p \models \neg\varphi$ if $p \not\models \varphi$.

The set $SB(p)$ is defined as $SB(p) = \{\varphi \in L_{SB} \mid p \models \varphi\}$. We write $p \sqsubseteq_{SB} q$ iff $SB(p) \subseteq SB(q)$ and we write $p =_{SB} q$ iff $SB(p) = SB(q)$. The negation operator collapses the preorder $\sqsubseteq_{SB}$ to $=_{SB}$ as the following proposition shows.

Proposition 8. *(Van Glabeek [3]).* $p \sqsubseteq_{SB} q \iff p =_{SB} q$.

Proof. If $\varphi \in SB(q) - SB(p)$ then $\neg\varphi \in SB(p) - SB(q)$. Hence, $p \sqsubseteq_{SB} q \iff p =_{SB} q$. $\square$

Theorem 9. *(Van Glabeek [3]).* $p \sim q \iff p =_{SB} q$.

Proof. ($\Longrightarrow$) By induction on the structure of φ. Since $p \sim q$, there exists a *strong bisimulation* R such that pRq.

- Let $p \models <a> \varphi$. Then there exists a $p' \in \mathbb{P}$ with $p \xrightarrow{a} p'$ and $p' \models \varphi$. Since pRq, there must be a $q' \in \mathbb{P}$ with $q \xrightarrow{a} q'$ and $p'Rq'$. So by induction $q' \models \varphi$ and hence $q \models <a> \varphi$.
  By symmetry, one also obtains $q \models <a> \varphi \Longrightarrow p \models <a> \varphi$.
- $p \models \bigwedge_{i\in I} \varphi_i \iff \forall i \in I\,(p \models \varphi_i) \overset{induction}{\iff} \forall i \in I\,(q \models \varphi_i) \iff q \models \bigwedge_{i\in I} \varphi_i$.
- $p \models \neg\varphi \iff p \not\models \varphi \overset{induction}{\iff} q \not\models \varphi \iff q \models \neg\varphi$.

($\Longleftarrow$) To prove that $p =_{SB} q \Longrightarrow p \sim q$, it suffices to establish that $=_{SB}$ is a strong bisimulation. If however, we show that $\sqsubseteq_{SB}$ is a strong simulation, then proposition 8 implies that $=_{SB} \;=\; \sqsubseteq_{SB} \;=\; \sqsubseteq_{SB}^{-1}$ is a strong bisimulation. We proceed to show that $\sqsubseteq_{SB}$ is a strong simulation.

Suppose, $p \sqsubseteq_{SB} q$ and $p \xrightarrow{a} p'$. Then $p \models <a> tt$ and $p \sqsubseteq_{SB} q$ implies q has at least one strong a-derivative. We have to show that $\exists q' \in \mathbb{P}$ with $q \xrightarrow{a} q'$ and $p' \sqsubseteq_{SB} q'$. Let

$$Q = \{q'' \in \mathbb{P} \mid q \xrightarrow{a} q'' \wedge p' \not\sqsubseteq_{SB} q''\}$$

For every $q'' \in Q$ there is a formula $\varphi_{q''} \in SB(p') - SB(q'')$. Now $<a> \bigwedge_{q''\in Q} \varphi_{q''} \in SB(p) \subseteq SB(q)$. Therefore there must be a $q' \in \mathbb{P}$ with $q \xrightarrow{a} q'$ and $q' \models \bigwedge_{q''\in Q} \varphi_{q''}$, which implies $q' \notin Q$. $\square$

With negation in the logic, we may define the duals of the operators in L_{SB} as derived ones.

$$[a]\varphi \equiv \neg <a> \neg\varphi \qquad\qquad \bigvee_{i\in \mathbf{I}} \varphi_i \equiv \neg \bigwedge_{i\in I} \neg\varphi_i$$

We may then define a negation-free language $L_{SB'}$.

Definition 10. The class $L_{SB'}$ of *negation-free strong bisimulation formulas* over Act is given by the following grammar

$$\varphi ::= <a> \varphi \mid \bigwedge_{i\in I} \varphi_i \mid [a]\varphi \mid \bigvee_{i\in I} \varphi_i$$

with $tt \equiv \bigwedge_{i\in\emptyset} \varphi_i$ and $ff \equiv \bigvee_{i\in\emptyset} \varphi_i$.

Definition 11. The *satisfaction relation* $\models \subseteq \mathbb{P} \times L_{SB'}$ is defined recursively as in the case of L_{SB} with the clause for negation being replaced by the following clauses.

- $p \models ff$ for no $p \in \mathbb{P}$.
- $p \models [a]\varphi$ for $a \in Act$ if $\forall\, p' \in \mathbb{P} : p \xrightarrow{a} p' \implies p' \models \varphi$.
- $p \models \bigvee_{i\in I} \varphi_i$ if $p \models \varphi_i$ for some $i \in I$.

It is clear that $\neg\neg\varphi$ is equivalent to φ. It may be shown (refer [2]) that L_{SB} and $L_{SB'}$ are expressively equivalent and characterize the same equivalence.

1.2 Weak Bisimulation

We may define several other derived operators. Some of the relevant ones are given below.

$<a>^0\, \varphi \equiv \varphi$	$<a>^{m+1}\, \varphi \equiv <a><a>^m\, \varphi, m \in \mathbb{N}$
$\ll\gg \varphi \equiv \bigvee_{m\geq 0} <\tau>^m\, \varphi$	$\ll a \gg \varphi \equiv \ll\gg<a>\ll\gg \varphi$
$[[\ ]]\varphi \equiv \neg \ll\gg \neg\varphi$	$[[a]]\varphi \equiv [[\ ]][a][[\ ]]\varphi$

The last four operators correspond to the weak transition relation $\Longrightarrow$ which is the smallest relation on processes such that

- $p \Longrightarrow p$ for all processes p, and
- $p \xrightarrow{\tau} q$ and $q \Longrightarrow r$ implies $p \Longrightarrow r$

Further for any $a \in Act$,

- $p \stackrel{a}{\Longrightarrow} q$ if $p \Longrightarrow \xrightarrow{a} \Longrightarrow q$, and q is called a *weak a-derivative* or simply an *a-derivative* of p.
- $p \stackrel{\hat{a}}{\Longrightarrow} q$ denotes $p \Longrightarrow q$ if $a = \tau$ and $p \stackrel{a}{\Longrightarrow} q$ otherwise.
- $\ll \hat{a} \gg$ denotes $\ll\gg$ if $a = \tau$ and $\ll a \gg$ otherwise.

Definition 12. A *binary relation* $R \subseteq \mathbb{P} \times \mathbb{P}$ is a **weak simulation** if for every $\langle p, q\rangle \in R$ and $a \in Act$,

$$p \xrightarrow{a} p' \implies \exists q' : q \stackrel{\hat{a}}{\Longrightarrow} q' \wedge p'Rq'$$

R is a **weak bisimulation (WB)** if both R and R^{-1} are weak simulations. We write $p \approx q$ if there exists a weak bisimulation R such that pRq.

Definition 13. The class L_{WB} of *weak bisimulation formulae* over Act is defined by the following BNF.

$$\varphi ::= \ \ll\alpha\gg \varphi \mid \bigwedge_{i\in I} \varphi_i \mid \neg\varphi$$

where $\alpha \in V$.

Note that in the light of our discussion on derived operators, L_{WB} is entirely contained in L_{SB}. It is also less expressive in the sense that it cannot express properties relating to the number of invisible moves a process might be able to make.

Definition 14. The *satisfaction relation* $\models \subseteq \mathbb{P} \times L_{WB}$ is defined recursively by:

- $p \models tt$ for all $p \in \mathbb{P}$.
- $p \models \ll a \gg \varphi$ for $a \in V$ if $\exists p' : p \stackrel{a}{\Longrightarrow} p' \wedge p' \models \varphi$
- $p \models \bigwedge_{i\in I} \varphi_i$ if $p \models \varphi_i$ for all $i \in I$.
- $p \models \neg\varphi$ if $p \not\models \varphi$.

The set of all formulas that a process p satisfies is defined as $WB(p) = \{\varphi \in L_{WB} \mid p \models \varphi\}$. We write $p \sqsubseteq_{WB} q$ iff $WB(p) \subseteq WB(q)$ and we write $p =_{WB} q$ iff $WB(p) = WB(q)$. The proofs that $p \sqsubseteq_{WB} q$ implies $p =_{WB} q$ and that $p \approx q$ iff $p =_{WB} q$ (i.e. that L_{WB} characterizes weak bisimilarity) are similar to the corresponding proofs for strong bisimilarity and hence are omitted. It is also possible to define a negation-free logic $L_{WB'}$ (analogous to $L_{SB'}$) which characterizes weak bisimilarity.

References

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3. R. J. van Glabeek. The linear time - branching time spectrum I. the semantics of concrete, sequential processes. In *Handbook of Process Algebra.* Elsevier Science B. V., Netherlands., 2001.