COL333/671: Introduction to AI

Semester I, 2021

Probabilistic Reasoning over Time

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Outline

- Last Class
  - Probabilistic Reasoning
- This Class
  - Probabilistic Reasoning over Time
- Reference Material
  - AIMA Ch. 15

Acknowledgement

These slides are intended for teaching purposes only. Some material has been used/adapted from web sources and from slides by Doina Precup, Dorsa Sadigh, Percy Liang, Mausam, Dan Klein, Anca Dragan, Nicholas Roy and others.

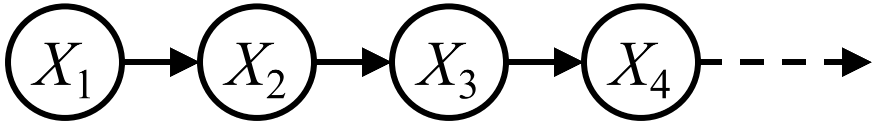
Reasoning: Sequence of Observations

- Reasoning over time or space

- Applications
  - Monitoring a disease
  - Robot localization
  - Target Tracking
  - Speech recognition
  - User attention
  - Gesture recognition

Markov Models

- Value of X at a given time is called the **state**.



$$P(X_1) \qquad P(X_t|X_{t-1})$$

- **Transition probabilities** or dynamics,
  - Specify how the state evolves over time
  - Initial state probabilities
- Stationarity assumption: transition probabilities the same at all times.
- (First order) Markov Property
  - Past and future independent given the present
  - Each time step only depends on the previous

Markov Models

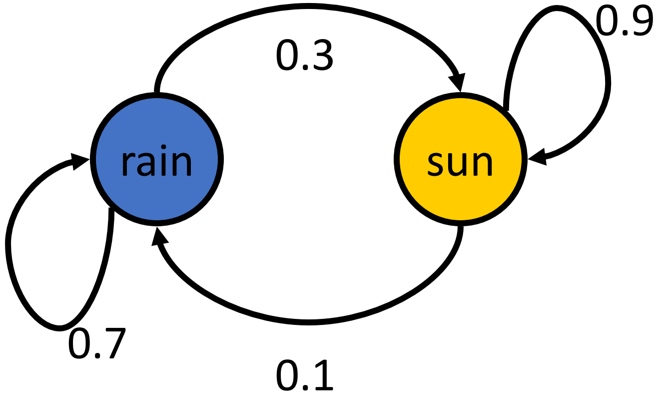
States: X = {rain, sun}

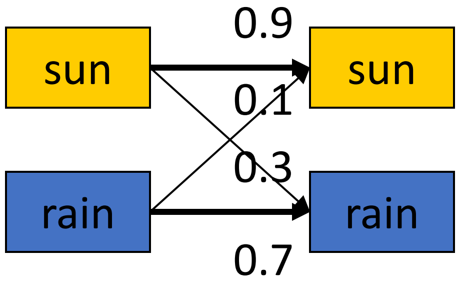
Initial distribution: 1.0 sun

CPT $P(X_t \mid X_{t-1})$:

X_{t-1}	X_t	$P(X_t \mid X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

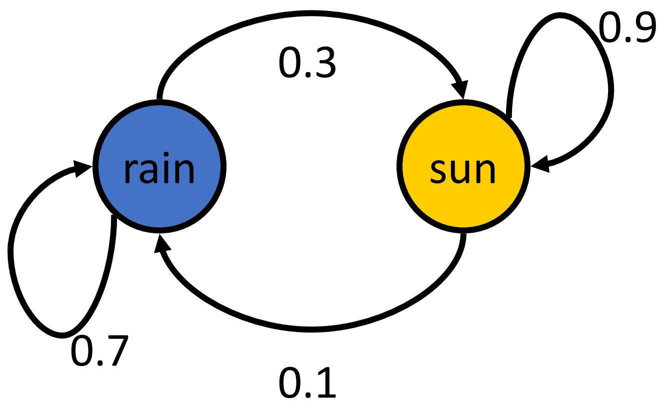
Representing the Markov model





Markov Models: Example

- Initial distribution: 1.0 sun



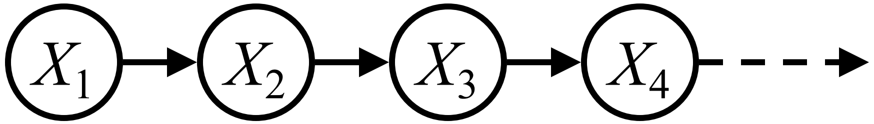
- What is the probability distribution after one step?

$$P(X_2 = \text{sun}) = P(X_2 = \text{sun} | X_1 = \text{sun}) P(X_1 = \text{sun}) + P(X_2 = \text{sun} | X_1 = \text{rain}) P(X_1 = \text{rain})$$

$$0.9 \cdot 1.0 + 0.3 \cdot 0.0 = 0.9$$

Forward Algorithm for a Markov Chain

- What's P(X) on some day t?



$$P(x_1) = \text{known}$$

$$P(x_t) = \sum_{x_{t-1}} P(x_{t-1}, x_t)$$

$$= \sum_{x_{t-1}} P(x_t \mid x_{t-1}) P(x_{t-1})$$

Forward simulation

Forward Algorithm for a Markov Chain

- From initial observation of sun

$$\left\langle \begin{matrix} 1.0 \\ 0.0 \end{matrix} \right\rangle \quad \left\langle \begin{matrix} 0.9 \\ 0.1 \end{matrix} \right\rangle \quad \left\langle \begin{matrix} 0.84 \\ 0.16 \end{matrix} \right\rangle \quad \left\langle \begin{matrix} 0.804 \\ 0.196 \end{matrix} \right\rangle \Rightarrow \left\langle \begin{matrix} 0.75 \\ 0.25 \end{matrix} \right\rangle$$

$$\mathrm{P}(X_1) \quad \mathrm{P}(X_2) \quad \mathrm{P}(X_3) \quad \mathrm{P}(X_4) \quad \mathrm{P}(X_\infty)$$

- From initial observation of rain

$$\left\langle \begin{matrix} 0.0 \\ 1.0 \end{matrix} \right\rangle \quad \left\langle \begin{matrix} 0.3 \\ 0.7 \end{matrix} \right\rangle \quad \left\langle \begin{matrix} 0.48 \\ 0.52 \end{matrix} \right\rangle \quad \left\langle \begin{matrix} 0.588 \\ 0.412 \end{matrix} \right\rangle \Rightarrow \left\langle \begin{matrix} 0.75 \\ 0.25 \end{matrix} \right\rangle$$

$$\mathrm{P}(X_1) \quad \mathrm{P}(X_2) \quad \mathrm{P}(X_3) \quad \mathrm{P}(X_4) \quad \mathrm{P}(X_\infty)$$

- From yet another initial distribution $\mathrm{P}(X_1)$:

$$\left\langle \begin{matrix} p \\ 1-p \end{matrix} \right\rangle \quad \cdots \quad \Rightarrow \left\langle \begin{matrix} 0.75 \\ 0.25 \end{matrix} \right\rangle$$

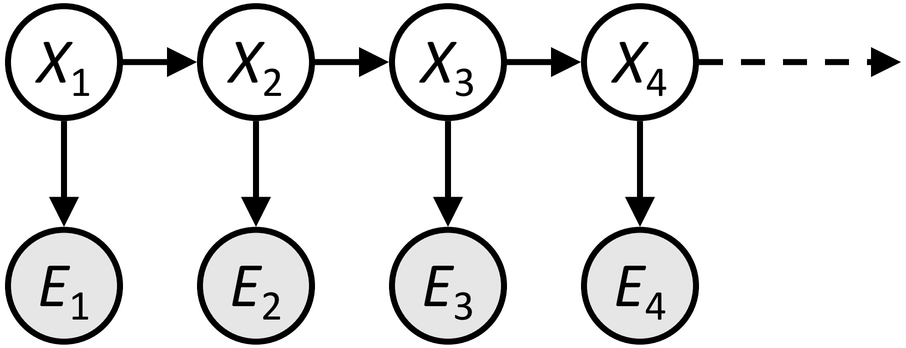
$$\mathrm{P}(X_1) \qquad \mathrm{P}(X_\infty)$$

Stationary distribution

$$P_\infty(X) = P_{\infty+1}(X) = \sum_x P(X|x)P_\infty(x)$$

Hidden Markov Models (HMMs)

- Markov Chains
  - Assume that we observe the state directly.
  - Often this is not the case. We only have noisy observations of the state.



- **Hidden** Markov Models
  - Underlying Markov chain over states X
  - You observe outputs (effects) at each time step

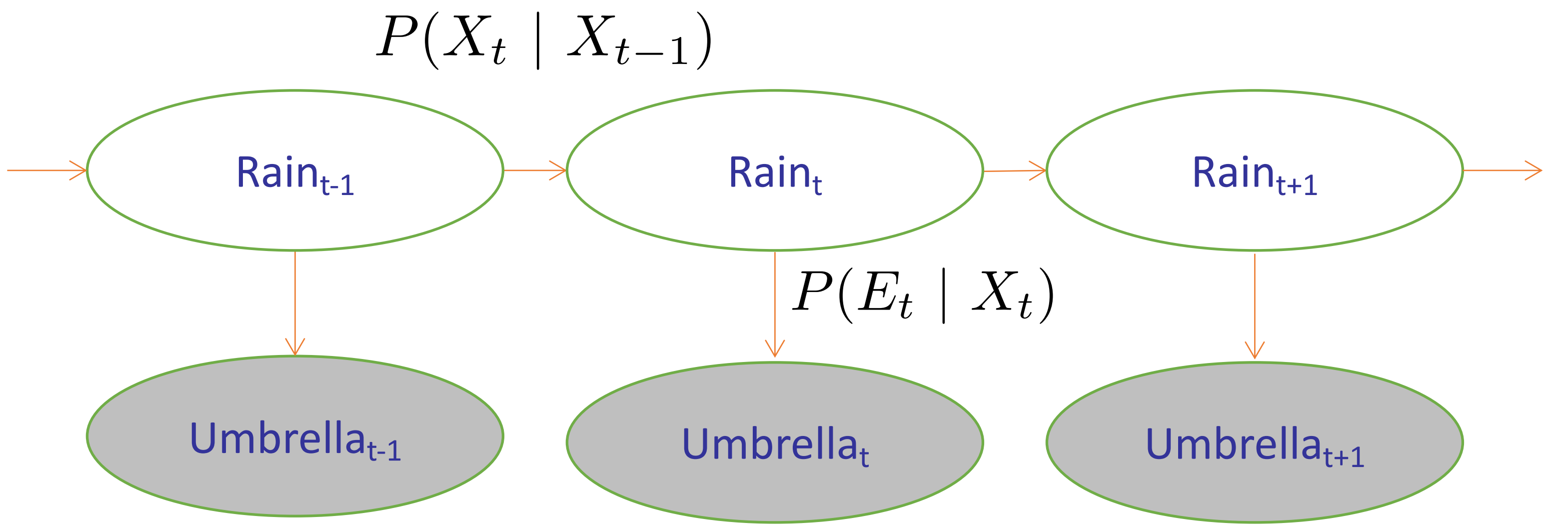
Weather HMM

The world state (rainy or sunny) is not directly observed. Instead have some observation such as a person carrying an umbrella or not.

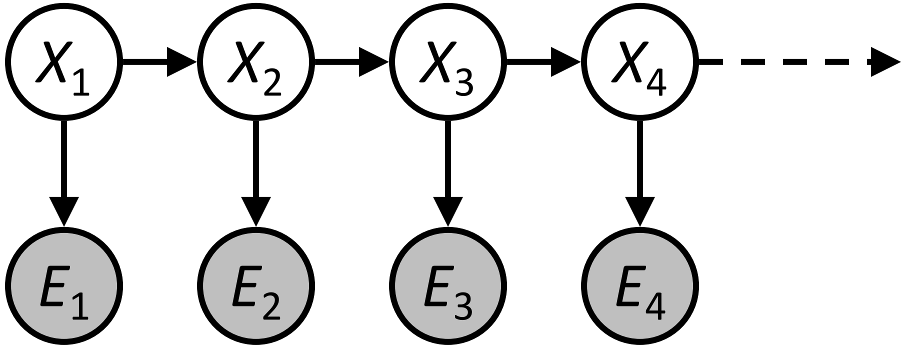
- An HMM is defined by:
  - Initial distribution: $P(X_1)$
  - Transitions: $P(X_t \mid X_{t-1})$
  - Emissions: $P(E_t \mid X_t)$

R_{t-1}	R_t	$P(R_t \mid R_{t-1})$
+r	+r	0.7
+r	-r	0.3
-r	+r	0.3
-r	-r	0.7

R_t	U_t	$P(U_t \mid R_t)$
+r	+u	0.9
+r	-u	0.1
-r	+u	0.2
-r	-u	0.8



HMMs – Conditional Independences



- Future depends on past via the present

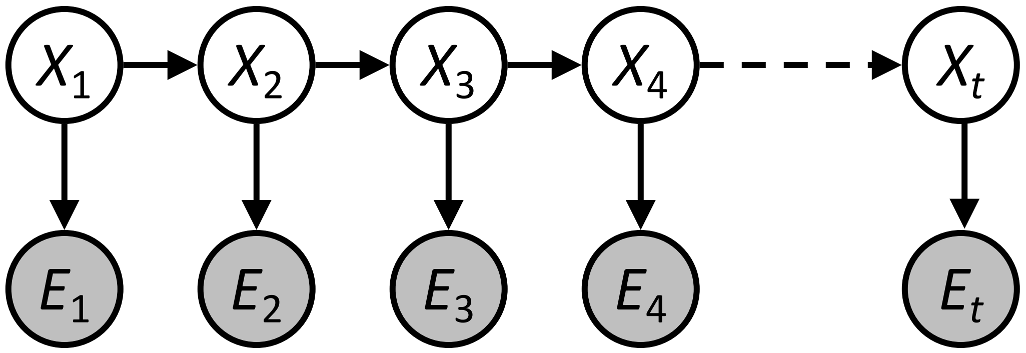
$$\mathbf{P}(\mathbf{X}_t \mid \mathbf{X}_{0:t-1}) = \mathbf{P}(\mathbf{X}_t \mid \mathbf{X}_{t-1})$$

- Current observation independent of all else given current state

$$\mathbf{P}(\mathbf{E}_t \mid \mathbf{X}_{0:t}, \mathbf{E}_{0:t-1}) = \mathbf{P}(\mathbf{E}_t \mid \mathbf{X}_t)$$

Example/Monitoring

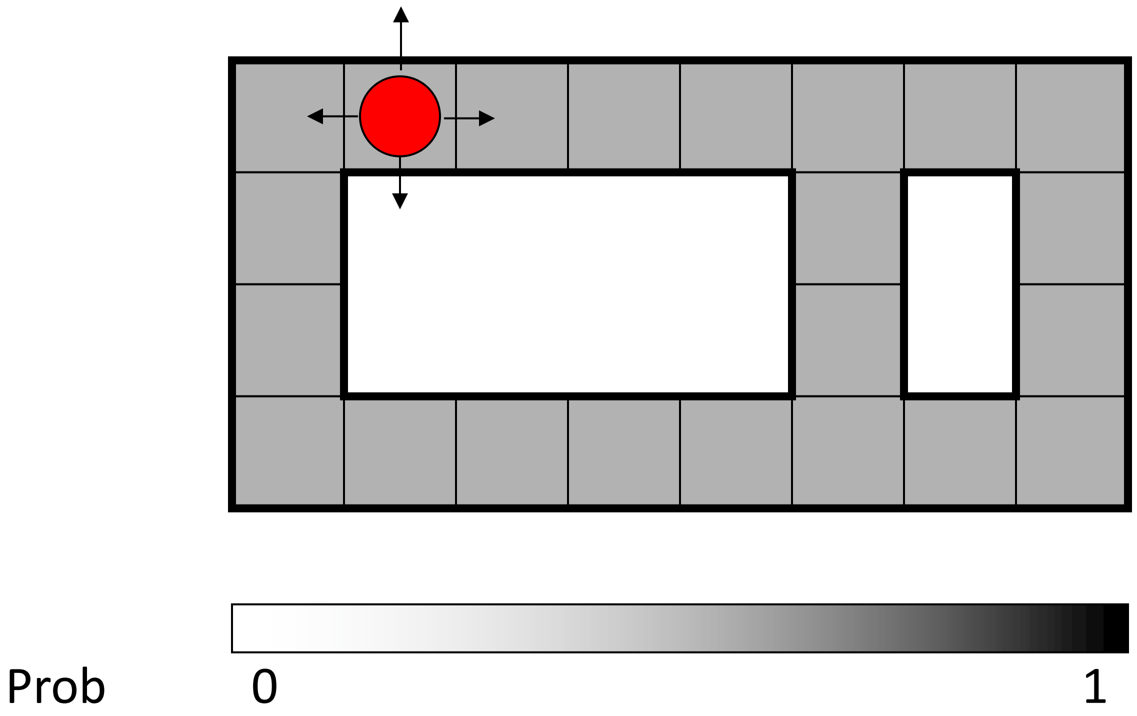
- Filtering, or monitoring, is the task of tracking the distribution
  - $B_t(X) = P_t(X_t \mid e_1, \ldots, e_t)$ (the belief state) over time



- We start with $B_1(X)$ in an initial setting, usually uniform

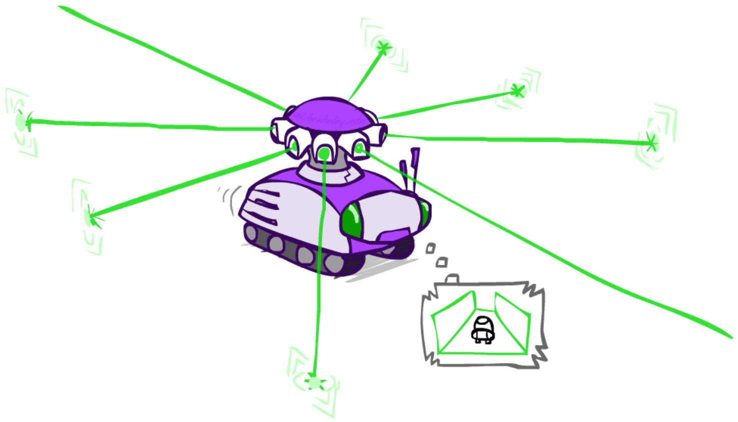
- As time passes, or we get observations, we update B(X)

Example: Robot Localization



t=0

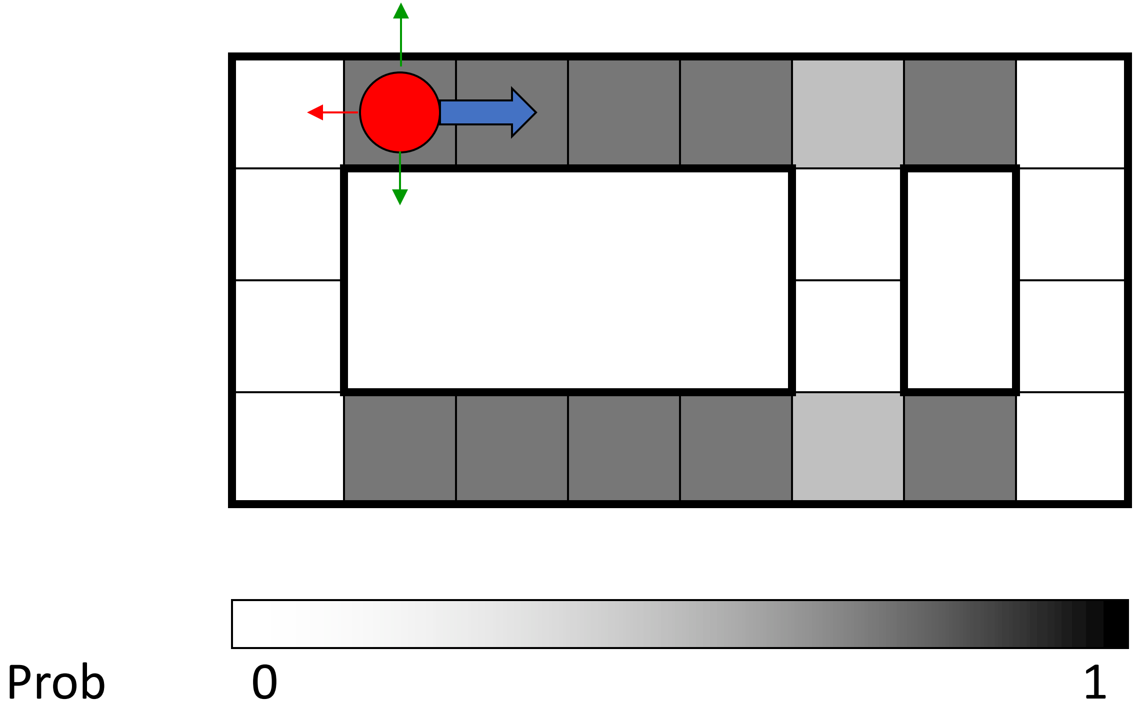
Robot can take actions N, S, E, W
Detects walls from its sensors

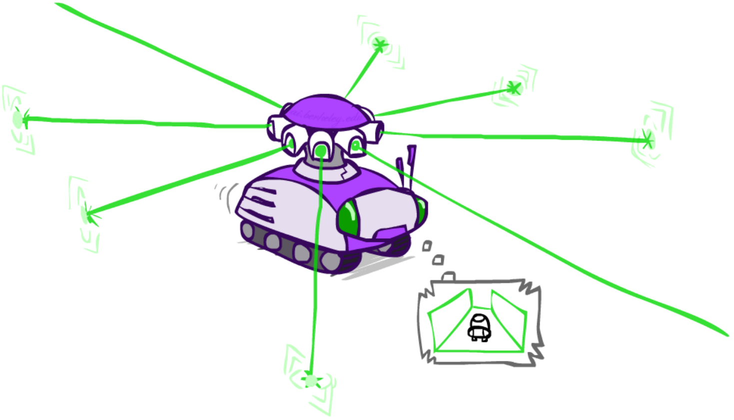


Sensor model: can read in which directions there is a wall, never more than 1 mistake

Motion model: may not execute action with small prob.

Example: Robot Localization

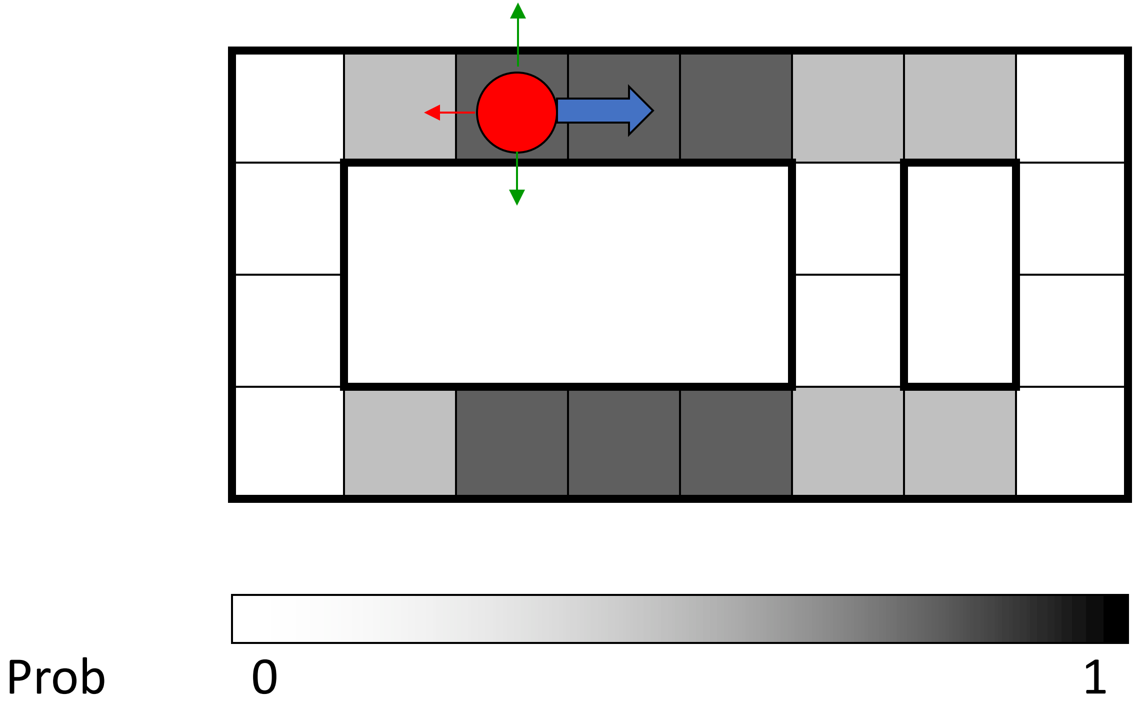




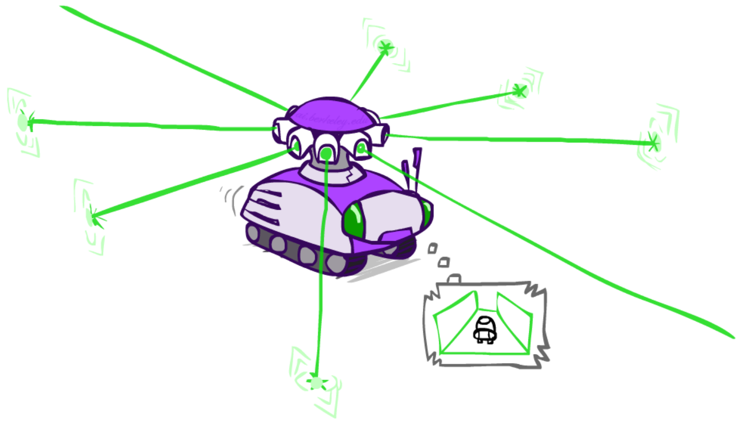
t=1

Lighter grey: was possible to get the reading, but less likely b/c required 1 mistake

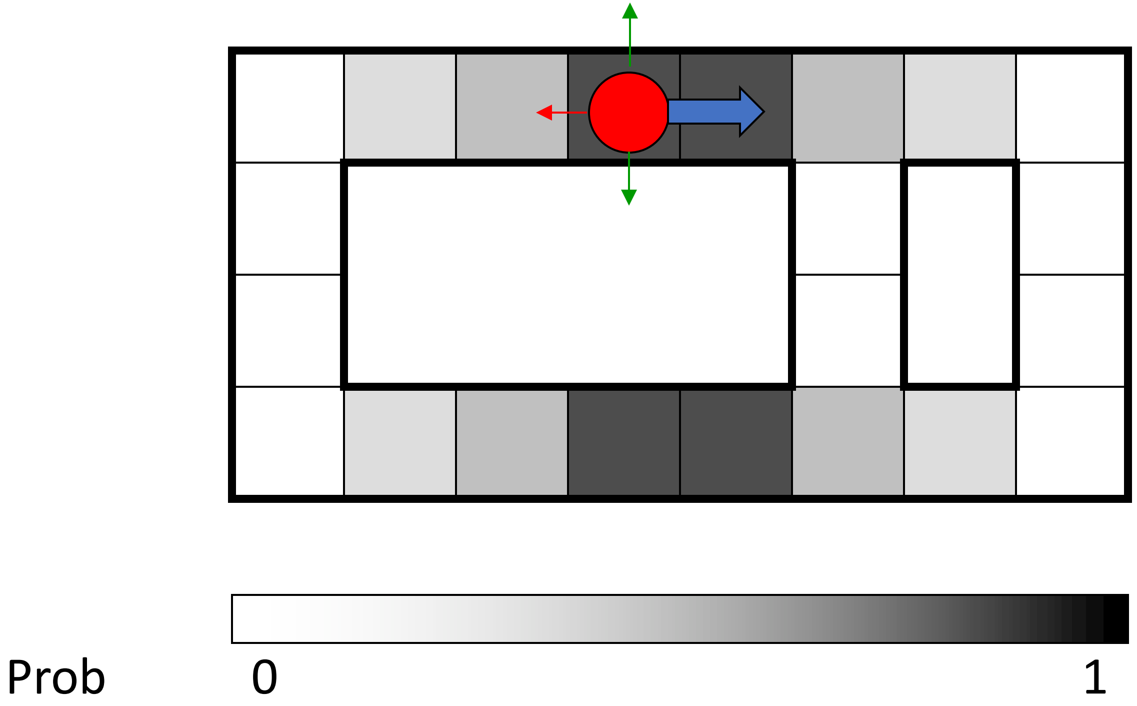
Example: Robot Localization



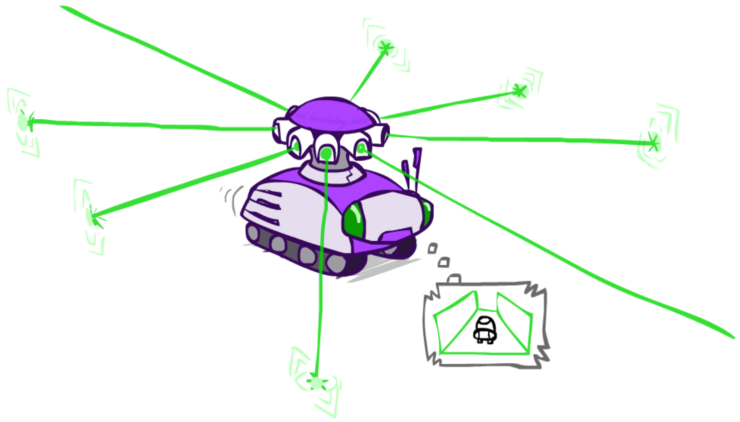
t=2



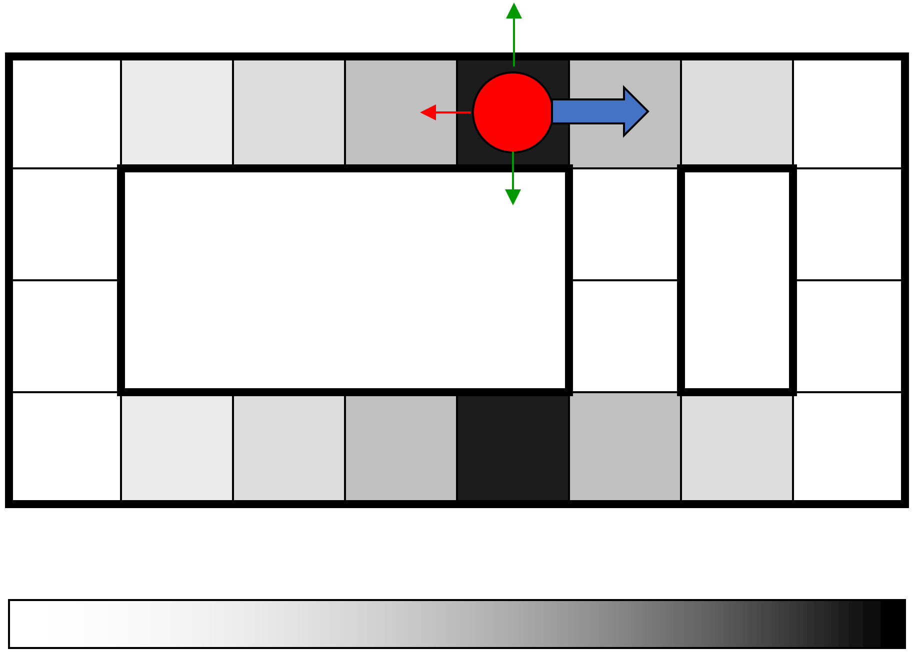
Example: Robot Localization



t=3

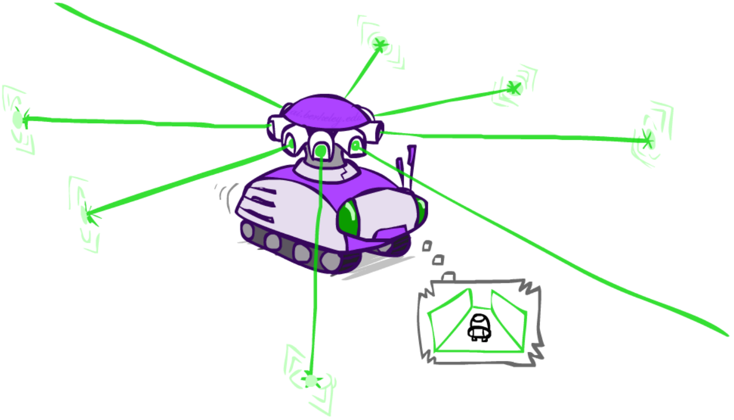


Example: Robot Localization

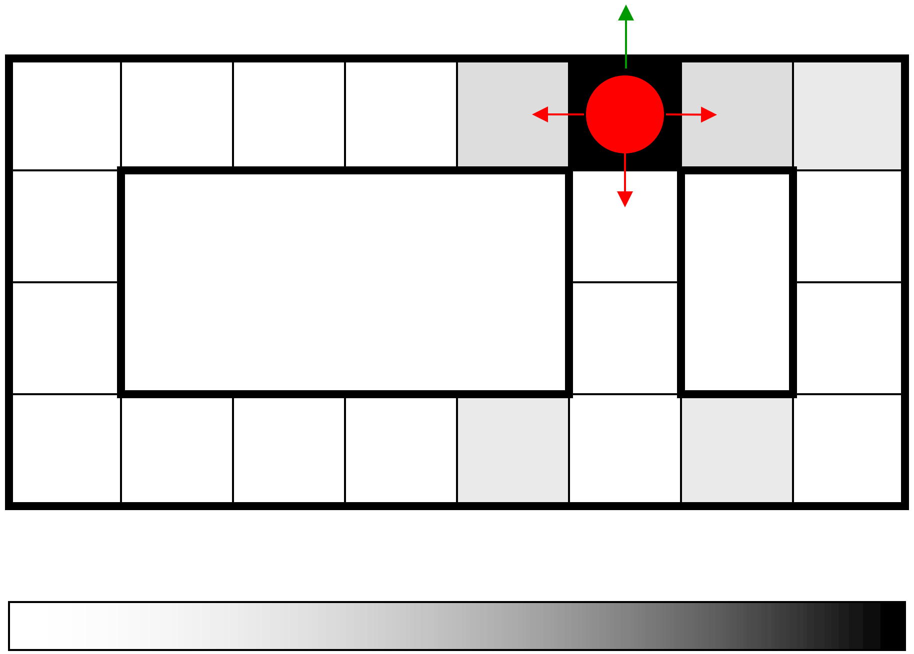


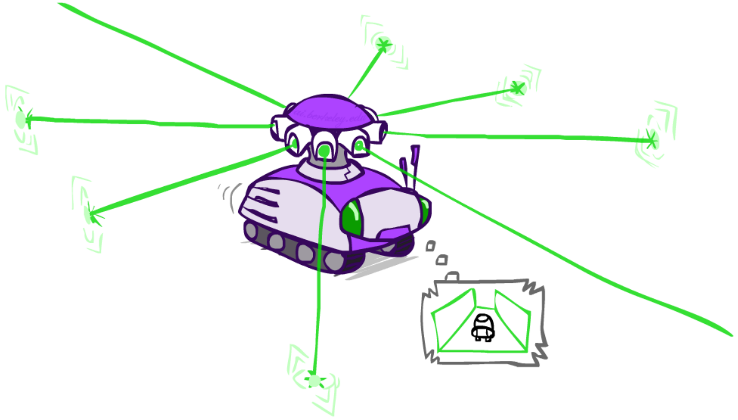
Prob 0 1

t=4



Example: Robot Localization





t=5

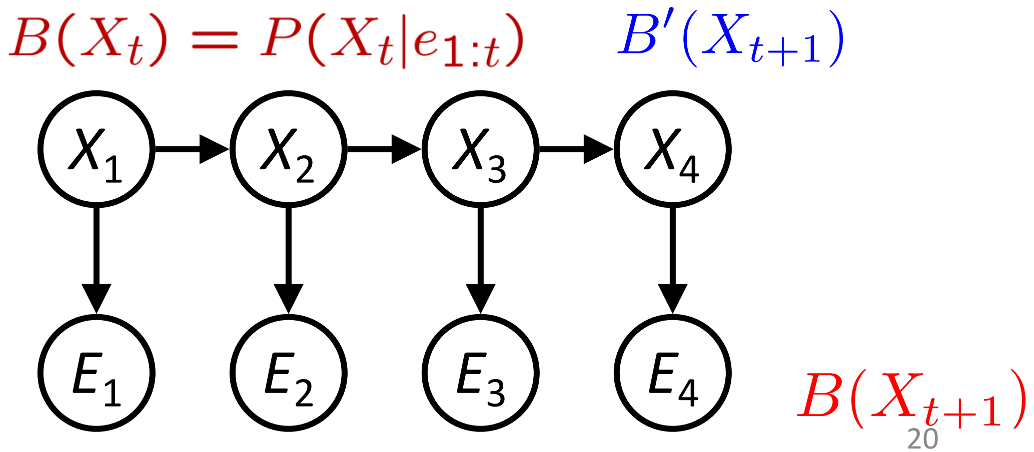
Inference: Estimate State Given Evidence

- We are given evidence at each time and want to know

$$B_t(X) = P(X_t|e_{1:t})$$

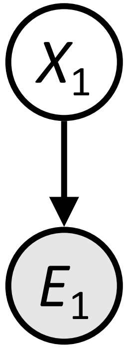
- Approach: start with $P(X_1)$ and derive B_t in terms of B_{t-1}
  - Equivalently, derive B_{t+1} in terms of B_t

- Two Steps:
  - Passage of time
  - Evidence incorporation



Estimating State Given Evidence: Base Cases

- Evidence incorporation
  - Incorporating noisy observations of the state.

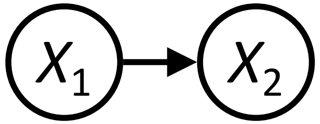


$$P(X_1|e_1)$$

$$P(X_1|e_1) = \frac{P(X_1, e_1)}{\sum_{x_1} P(x_1, e_1)}$$

$$P(X_1|e_1) = \frac{P(e_1|X_1)P(X_1)}{\sum_{x_1} P(e_1|x_1)P(x_1)}$$

- Passage of time
  - The system state at the next time step given transition model



$$P(X_2)$$

$$P(X_2) = \sum_{x_1} P(x_1, X_2)$$

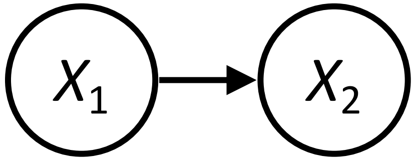
$$P(X_2) = \sum_{x_1} P(X_2|x_1)P(x_1)$$

Next, perform these two computations repeatedly over each time step

Passage of Time

Assume we have current belief P(X | evidence to date)

$$B(X_t) = P(X_t | e_{1:t})$$



Then, after one time step:

$$P(X_{t+1} | e_{1:t}) = \sum_{x_t} P(X_{t+1}, x_t | e_{1:t})$$

$$= \sum_{x_t} P(X_{t+1} | x_t, e_{1:t}) P(x_t | e_{1:t})$$

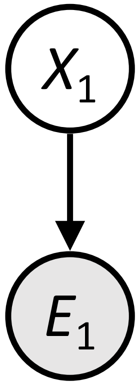
$$= \sum_{x_t} P(X_{t+1} | x_t) P(x_t | e_{1:t})$$

Basic idea: the beliefs get "pushed" through the transitions

Incorporating Observations

Assume we have current belief P(X | previous evidence):

$$B'(X_{t+1}) = P(X_{t+1}|e_{1:t})$$



Then, after evidence comes in:

$$P(X_{t+1}|e_{1:t+1}) = P(X_{t+1}, e_{t+1}|e_{1:t})/P(e_{t+1}|e_{1:t})$$

$$\propto_{X_{t+1}} P(X_{t+1}, e_{t+1}|e_{1:t})$$

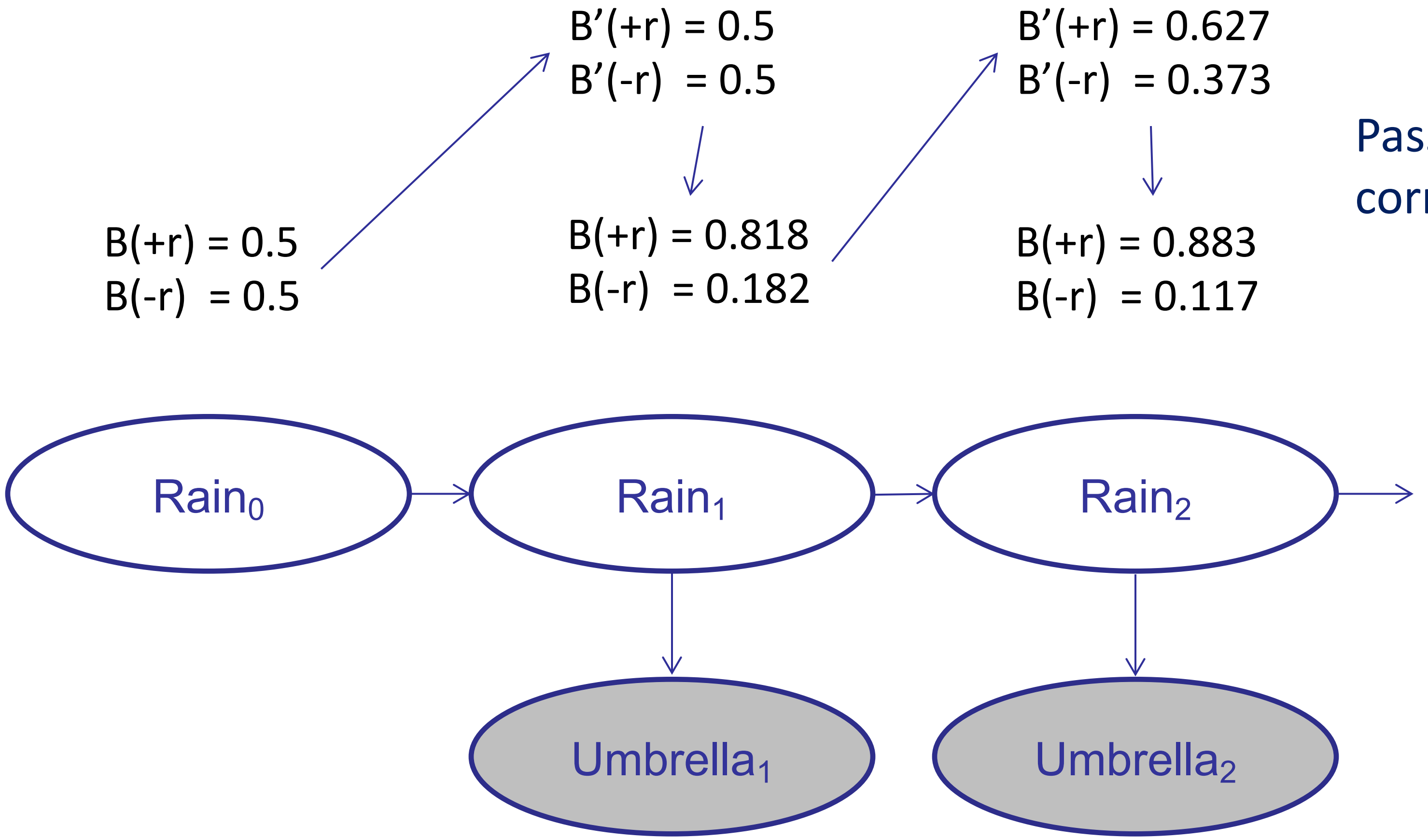
$$= P(e_{t+1}|e_{1:t}, X_{t+1})P(X_{t+1}|e_{1:t})$$

$$= P(e_{t+1}|X_{t+1})P(X_{t+1}|e_{1:t})$$

View it as a "correction" of the belief using the observation

$$B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1}|X_{t+1})B'(X_{t+1})$$

Inference: Weather HMM



Passage of time and correction at each stage.

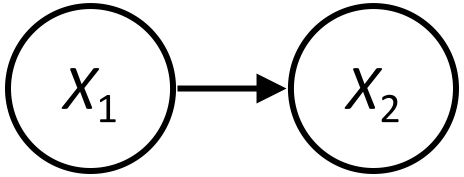
R_t	R_{t+1}	$P(R_{t+1}\|R_t)$
+r	+r	0.7
+r	-r	0.3
-r	+r	0.3
-r	-r	0.7

R_t	U_t	$P(U_t\|R_t)$
+r	+u	0.9
+r	-u	0.1
-r	+u	0.2
-r	-u	0.8

Online Belief Updates: Inference over Time

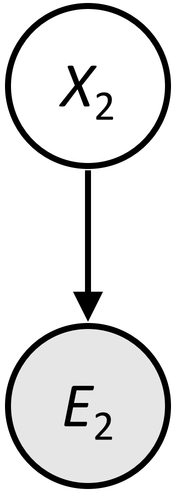
- Every time step, we start with current P(X | evidence)
- We update for time:

$$P(x_t|e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1}|e_{1:t-1}) \cdot P(x_t|x_{t-1})$$



- We update for evidence:

$$P(x_t|e_{1:t}) \propto_X P(x_t|e_{1:t-1}) \cdot P(e_t|x_t)$$



Forward Algorithm

We are given evidence at each time and want to know

$$B_t(X) = P(X_t | e_{1:t})$$

We can derive the following updates

$$P(x_t | e_{1:t}) \propto_{X_t} P(x_t, e_{1:t})$$

$$= \sum_{x_{t-1}} P(x_{t-1}, x_t, e_{1:t})$$

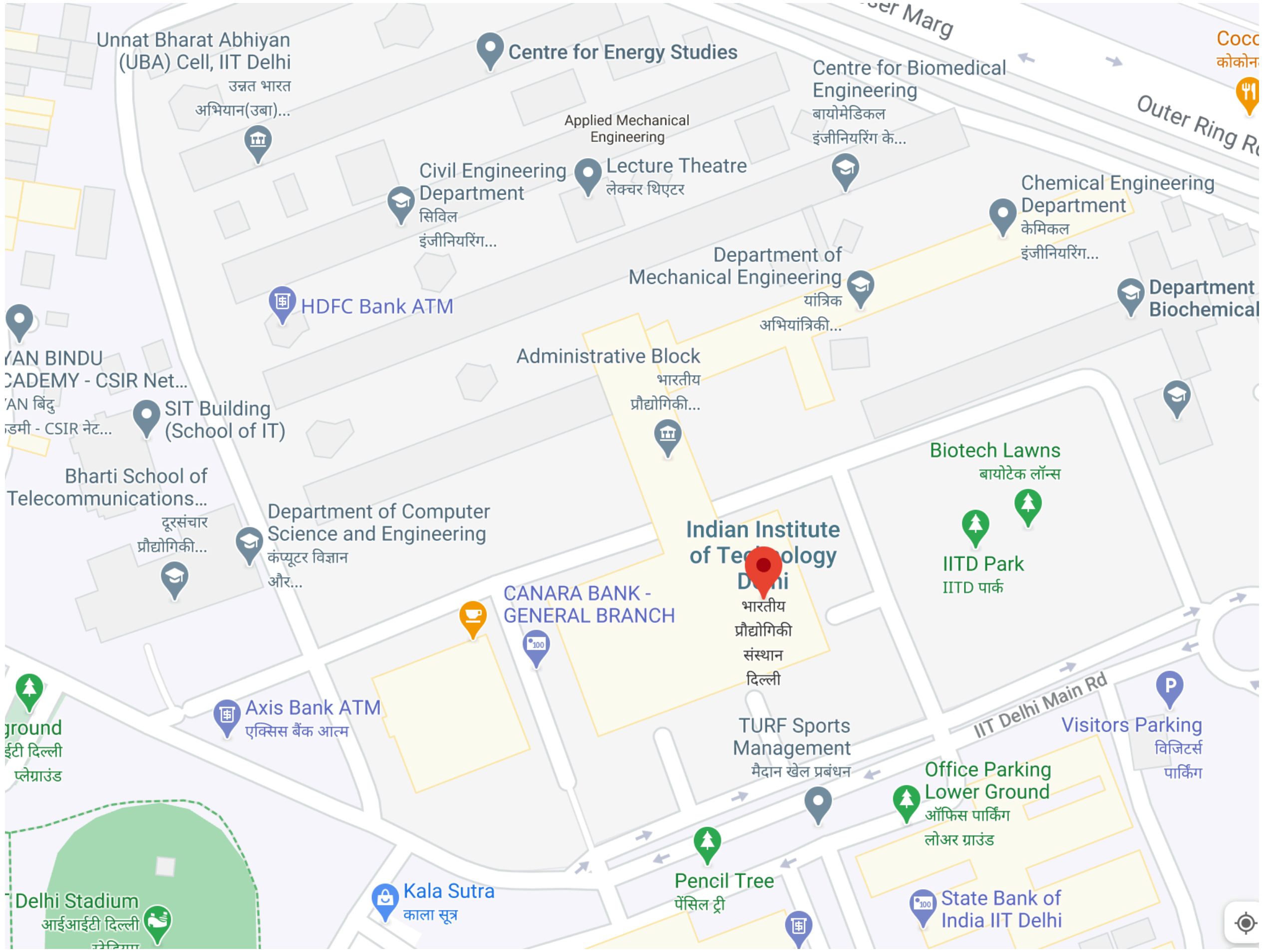
$$= \sum_{x_{t-1}} P(x_{t-1}, e_{1:t-1}) P(x_t | x_{t-1}) P(e_t | x_t)$$

$$= P(e_t | x_t) \sum_{x_{t-1}} P(x_t | x_{t-1}) P(x_{t-1}, e_{1:t-1})$$

Normalization can be at each step if the exact likelihood is needed at each step or at the end.

Large Number of States

A grid over a large space can lead to a large state space.



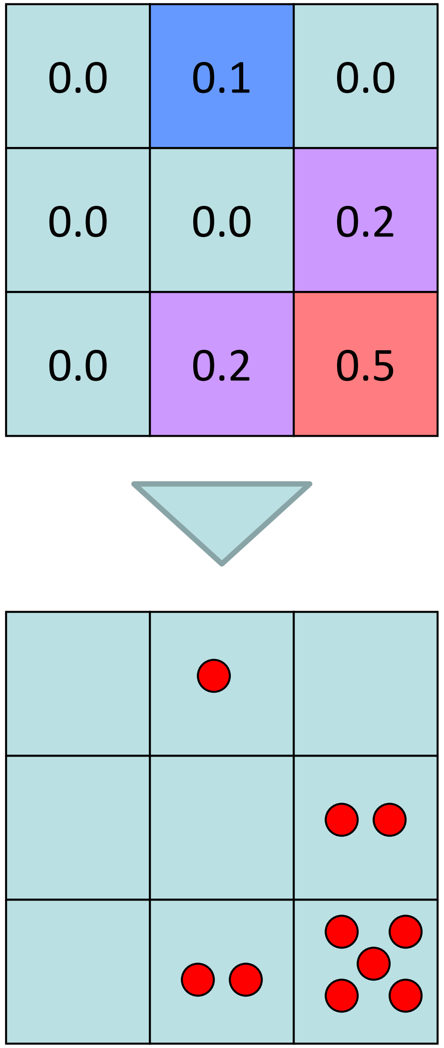
Particle Filtering

Problem: Sometimes |X| is too big to use exact inference

- |X| may be too big to even store B(X)
- E.g. X is continuous (though here we focus on the discrete case)

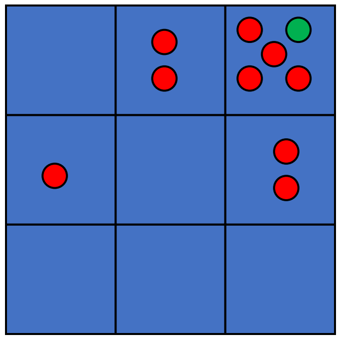
Solution: approximate inference

- Track samples of X, not all values.
- Samples are called "particles"
- Time spent per step is linear in the number of samples
- Keep the list of particles in memory, not states
- Larger the number of particles, the better is the approximation.



Representation: Particles

- Our representation of P(X) is now a list of N particles (samples)
  - Generally, N << |X|
- P(x) approximated by number of particles with value x
  - Several x can have P(x) = 0. Note that (3,3) has half the number of particles.
  - Larger the number of particles, better is the approximation.



Particles:
(3,3)
(2,3)
(3,3)
(3,2)
(3,3)
(3,2)
(1,2)
(3,3)
(3,3)
(2,3)

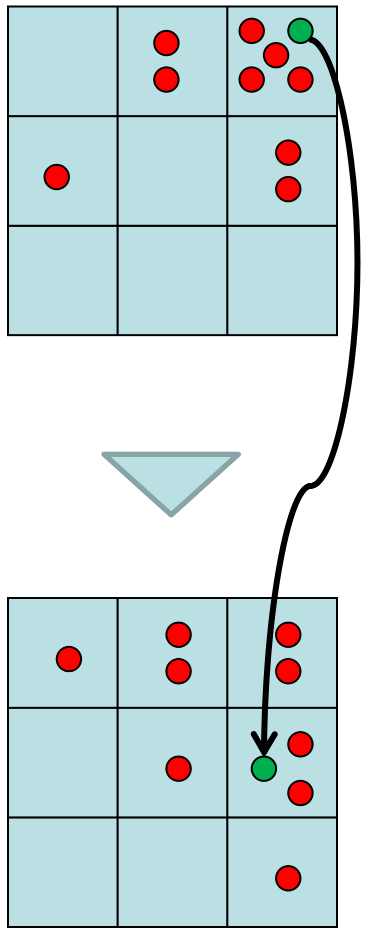
Representation: Passage of Time

Each particle is moved by sampling its next position from the transition model

$$x' = \text{sample}(P(X'|x))$$

- Perform simulation or sampling
  - The samples' frequencies reflect the transition probabilities
- In the example, most samples move clockwise, but some move in another direction or stay in place.
  - This is an outcome of the probabilistic transition model.

Particles:
(3,3)
(2,3)
(3,3)
(3,2)
(3,3)
(3,2)
(1,2)
(3,3)
(3,3)
(2,3)



Particles:
(3,2)
(2,3)
(3,2)
(3,1)
(3,3)
(3,2)
(1,3)
(2,3)
(3,2)
(2,2)

Representation: Incorporate Evidence

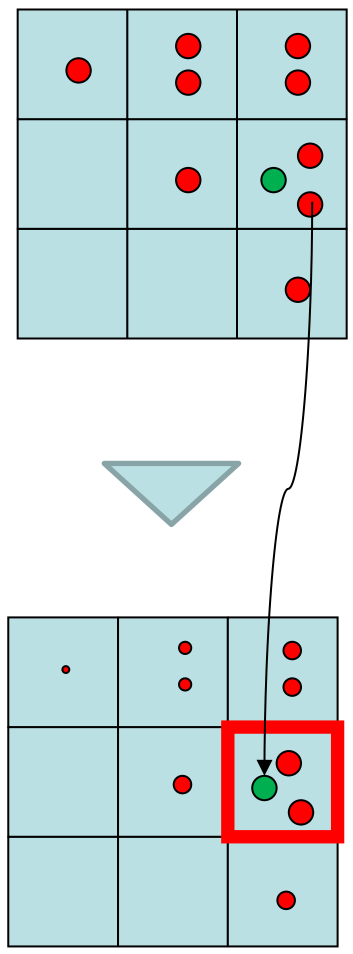
- As seen previously, incorporating evidence adjusts or weighs the probabilities.
- Attach a weight to each sample.
- Weigh the samples based on the likelihood of the evidence.

$$w(x) = P(e|x)$$

$$B(X) \propto P(e|X)B'(X)$$

Particles:
(3,2)
(2,3)
(3,2)
(3,1)
(3,3)
(3,2)
(1,3)
(2,3)
(3,2)
(2,2)

Particles:
(3,2) w=.9
(2,3) w=.2
(3,2) w=.9
(3,1) w=.4
(3,3) w=.4
(3,2) w=.9
(1,3) w=.1
(2,3) w=.2
(3,2) w=.9
(2,2) w=.4

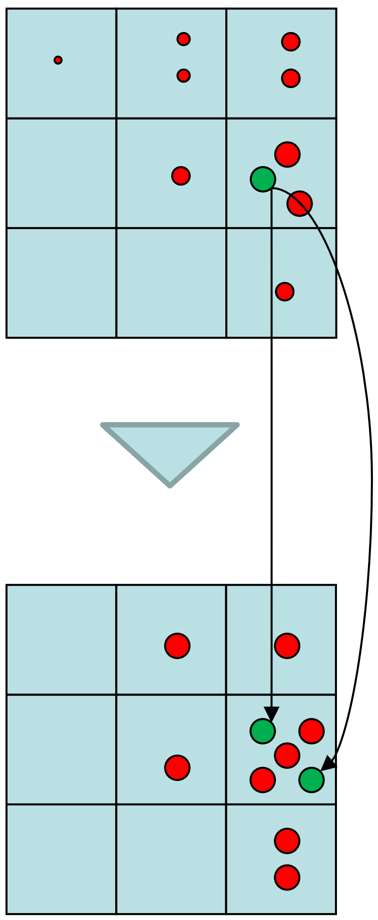


Representation: Resample

- Rather than tracking weighted samples, we resample
- N times, we choose from our weighted sample distribution (i.e. draw with replacement)
- Now the update is complete for this time step, continue with the next one

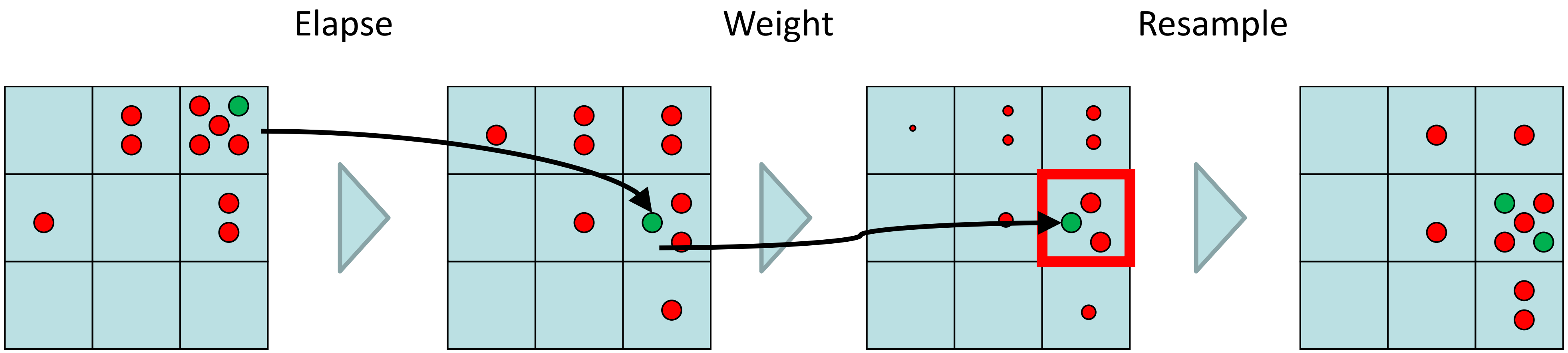
Particles:
(3,2) w=.9
(2,3) w=.2
(3,2) w=.9
(3,1) w=.4
(3,3) w=.4
(3,2) w=.9
(1,3) w=.1
(2,3) w=.2
(3,2) w=.9
(2,2) w=.4

(New) Particles:
(3,2)
(2,2)
(3,2)
(2,3)
(3,3)
(3,2)
(1,3)
(2,3)
(3,2)
(3,2)



Representation: Particles

Particles: track samples of states rather than an explicit distribution

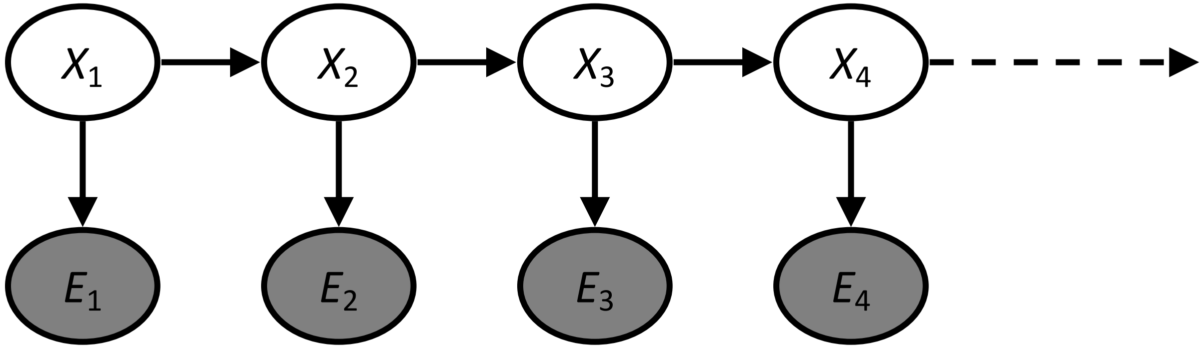


Particles:	Particles:	Particles:	(New) Particles:
(3,3)	(3,2)	(3,2) w=.9	(3,2)
(2,3)	(2,3)	(2,3) w=.2	(2,2)
(3,3)	(3,2)	(3,2) w=.9	(3,2)
(3,2)	(3,1)	(3,1) w=.4	(2,3)
(3,3)	(3,3)	(3,3) w=.4	(3,3)
(3,2)	(3,2)	(3,2) w=.9	(3,2)
(1,2)	(1,3)	(1,3) w=.1	(1,3)
(3,3)	(2,3)	(2,3) w=.2	(2,3)
(3,3)	(3,2)	(3,2) w=.9	(3,2)
(2,3)	(2,2)	(2,2) w=.4	(3,2)

Most Likely Explanation

HMMs defined by

- States X
- Observations E
- Initial distribution: $P(X_1)$
- Transitions: $P(X|X_{-1})$
- Emissions: $P(E|X)$



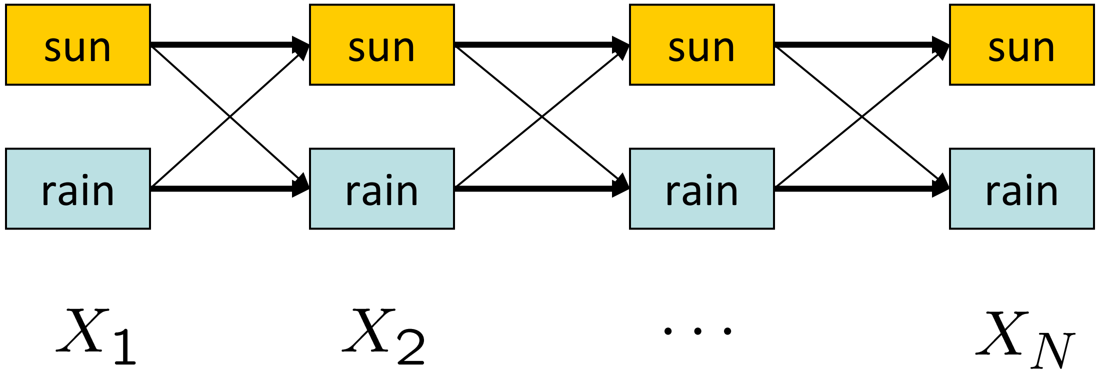
Problem: Most-likely Explanation $\arg\max_{x_{1:t}} P(x_{1:t}|e_{1:t})$

Determine the most likely sequence of states given all the evidence.

Solution: the Viterbi algorithm

State Trellis

State trellis: graph of states and transitions over time



Each arc represents some transition $x_{t-1} \rightarrow x_t$

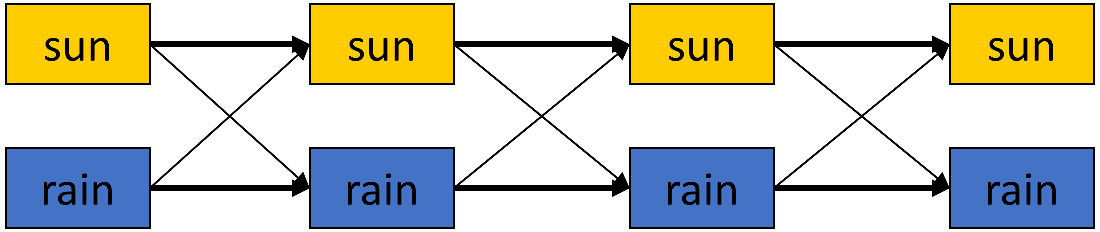
Each arc has weight $P(x_t|x_{t-1})P(e_t|x_t)$

Each path is a sequence of states

The product of weights on a path is that sequence's probability along with the evidence

Forward algorithm computes sums of paths, Viterbi computes best paths

Viterbi Algorithm



$$X_1 \quad X_2 \quad \cdots \quad X_N$$

Forward Algorithm (Sum)

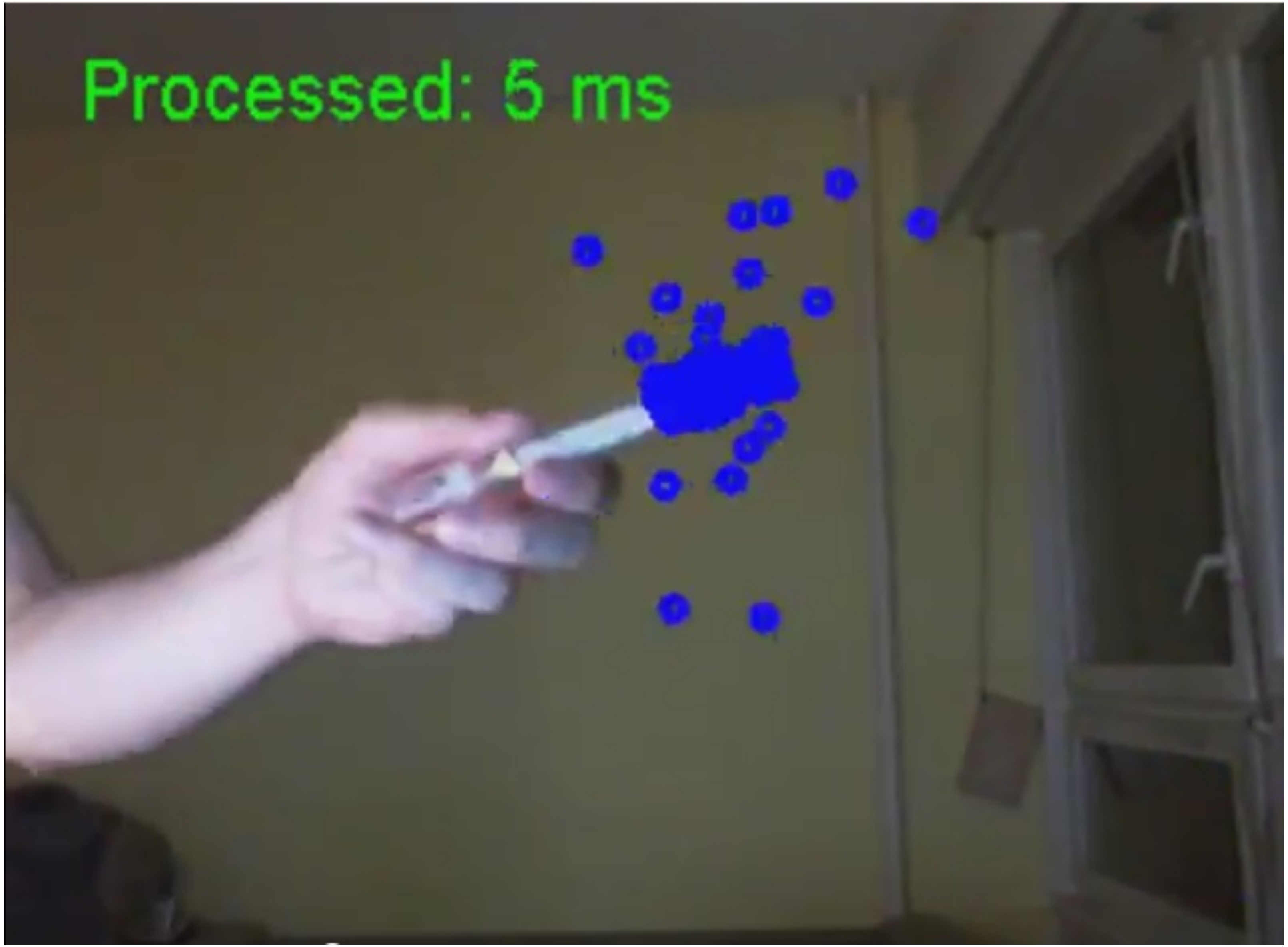
$$f_t[x_t] = P(x_t, e_{1:t})$$

$$= P(e_t|x_t) \sum_{x_{t-1}} P(x_t|x_{t-1}) f_{t-1}[x_{t-1}]$$

Viterbi Algorithm (Max)

$$m_t[x_t] = \max_{x_{1:t-1}} P(x_{1:t-1}, x_t, e_{1:t})$$

$$= P(e_t|x_t) \max_{x_{t-1}} P(x_t|x_{t-1}) m_{t-1}[x_{t-1}]$$



Application: tracking of a red pen. The blue dots indicate the estimated positions.
Video: https://www.youtube.com/watch?v=SV6CmEha51k