

1. Six swimmers training together either swam in a race or watched the others swim. At least how many races must have been scheduled if every swimmer had opportunity to watch all of the others?
2. Suppose that 21 girls and 21 boys enter a mathematics competition. Furthermore, suppose that each entrant solves at most six questions, and for every boy-girl pair, there is at least one question that they both solved. Show that there is a question that was solved by at least three girls and at least three boys.
3. How many ways are there for a horse race with four horses to finish if ties are possible?

Recall (section on Permutation and Combination from the book) that a *combinatorial proof* of an identity is a proof that uses counting arguments to prove that both sides of the identity count the same objects but in different ways or a proof that is based on showing that there is a bijection between the sets of objects counted by the two sides of the identity. These two types of proofs are called double counting proofs and bijective proofs, respectively.

4. Prove the *Hockeystick identity*

$$\sum_{k=0}^r \binom{n+k}{k} = \binom{n+r+1}{r}$$

whenever n and r are positive integers,

- (a) using a combinatorial argument.
 - (b) using Pascal's identity (i.e., $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$).
5. Give a combinatorial proof that $\sum_{k=1}^n k \binom{n}{k} = n2^{n-1}$.
 6. Give a combinatorial proof that $\sum_{k=1}^n k \binom{n}{k}^2 = n \binom{2n-1}{n-1}$.
 7. Suppose that a weapons inspector must inspect each of five different sites twice, visiting one site per day. The inspector is free to select the order in which to visit these sites, but cannot visit site X , the most suspicious site, on two consecutive days. In how many different orders can the inspector visit these sites?
 8. Suppose that instead of three doors, there are four doors in the Monty Hall puzzle. What is the probability that you win by not changing once the host, who knows what is behind each door, opens a losing door and gives you the chance to change doors? What is the probability that you win by changing the door you select to one of the two remaining doors among the three that you did not select?