

1. Prove or disprove: Any strongly connected graph with n and $(n - 1)$ edges is a tree.
2. Let S be the subset of the set of ordered pairs of integers defined recursively by:
Basis step: $(0, 0) \in S$
Recursive step: If $(a, b) \in S$, then $(a, b + 1) \in S$, $(a + 1, b + 1) \in S$, and $(a + 2, b + 1) \in S$.
 - (a) List the elements of S produced by the first four applications of the recursive definition.
 - (b) Use strong induction on the number of applications of the recursive step of the definition to show that $a \leq 2b$ whenever $(a, b) \in S$.
 - (c) Use structural induction to show that $a \leq 2b$ whenever $(a, b) \in S$.
3. A lattice point in the plane is a point (x, y) where both x and y are integers. Use mathematical induction to show that at least $n + 1$ straight lines are needed to ensure that every lattice point (x, y) with $x \geq 0$, $y \geq 0$, and $x + y \leq n$ lies on one of these lines.
4. Use structural induction to show that $l(T)$, the number of leaves of a full binary tree T , is 1 more than $i(T)$, the number of internal vertices of T .
5. Suppose there are n people in a group, each of whom is aware of a scandal that no one else in the group knows about. These people communicate by telephone; when two people in the group talk, they share information about all the scandals each knows about. For example, on the first call, two people share information, so by the end of the call, each person knows about two scandals. The gossip problem asks for $G(n)$, the minimum number of telephone calls that are needed for all n people to learn about all the scandals.
 1. Find $G(1)$, $G(2)$, $G(3)$, and $G(4)$.
 2. Use mathematical induction to prove that $G(n) \leq 2n - 4$ for $n \geq 4$.
Hint: In the inductive step, have a new person call a particular person at the start and at the end.
 3. Prove that $G(n) = 2n - 4$ for $n \geq 4$.