

1. Show that  $\log_2 3$  is an irrational number.
2. Prove or disprove that there are three consecutive odd positive integers that are primes, that is, odd primes of the form  $p$ ,  $p + 2$ , and  $p + 4$ .
3. The value of the Euler  $\phi$ -function at the positive integer  $n$  is defined to be the number of positive integers less than or equal to  $n$  that are relatively prime to  $n$ . For instance,  $\phi(6) = 2$  because of the positive integers less than or equal to 6, only 1 and 5 are relatively prime to 6.

Answer the following questions:

- (a) What are the values of  $\phi(4)$ ,  $\phi(10)$ ,  $\phi(13)$ ?
  - (b) What is the value of  $\phi(p^k)$  when  $p$  is a prime and  $k$  is a positive integer?
  - (c) What is the value of  $\phi(pq)$  when  $p$  and  $q$  are prime numbers.
4. Prove or disprove:  $n^2 - 79n + 1601$  is prime whenever  $n$  is a positive integer.
  5. Show that if  $a$  and  $m$  are relatively prime positive integers, then the inverse of  $a$  modulo  $m$  is unique modulo  $m$ .
  6. Find the solutions of the congruence  $15x^2 + 19x \equiv 5 \pmod{11}$ .
  7. Show that if  $p$  is prime, the only solutions of  $x^2 \equiv 1 \pmod{p}$  are integers  $x$  such that  $x \equiv 1 \pmod{p}$  or  $x \equiv -1 \pmod{p}$ .
  8. Show that if  $p$  is a prime, the positive integers less than  $p$ , except 1 and  $p - 1$ , can be split into  $(p - 3)/2$  pairs of integers such that each pair consists of integers that are inverses of each other.  
Use the above to show that  $(p - 1)! \equiv -1 \pmod{p}$ . This result is known as *Wilson's Theorem*.