

ACOL1002: Discrete Mathematical Structures

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Logic

Rules of inference: Propositional logic

Rule of inference	Tautology	Name
$\frac{p \quad p \rightarrow q}{\therefore q}$	$[p \wedge (p \rightarrow q)] \rightarrow q$	Modus ponens
$\frac{\neg q \quad p \rightarrow q}{\therefore \neg p}$	$[\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$	Modus tollens
$\frac{p \rightarrow q \quad q \rightarrow r}{\therefore p \rightarrow r}$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$\frac{p \vee q \quad \neg p}{\therefore q}$	$[(p \vee q) \wedge \neg p] \rightarrow (q)$	Disjunctive syllogism

Table: Rules of inference.

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Rules of inference: Propositional logic

Rule of inference	Tautology	Name
$\frac{p}{\therefore p \vee q}$	$p \rightarrow (p \vee q)$	Addition
$\frac{p \wedge q}{\therefore p}$	$(p \wedge q) \rightarrow p$	Simplification
$\frac{p}{q}$ $\frac{q}{\therefore p \wedge q}$	$[(p) \wedge (q)] \rightarrow (p \wedge q)$	Conjunction
$\frac{p \vee q}{\neg p \vee r}$ $\frac{\neg p \vee r}{\therefore q \vee r}$	$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$	Resolution

Table: Rules of inference.

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Rules of inference: Propositional logic

- Which rule of inference is used in the argument:
If it rains today, then we will not have a barbecue today.
If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have barbecue tomorrow.

Resolution Principle

- *Resolution Principle* is another way of showing that an argument is correct.
- Definitions:
 - Literal: A variable or a negation of a variable is called a literal.
 - Sum and Product: A disjunction of literals is called a sum and a conjunction of literals is called a product.
 - Clause: A disjunction of literals is called a clause.
 - Resolvent: For any two clauses C_1 and C_2 , if there is a literal L_1 in C_1 that is complementary to literal L_2 in C_2 , then delete L_1 and L_2 from C_1 and C_2 respectively and construct the disjunction of the remaining clauses. The constructed clause is a resolvent of C_1 and C_2 .
 - $C_1 = P \vee Q \vee R$
 - $C_2 = \neg P \vee \neg S \vee T$
 - What is a resolvent of C_1 and C_2 ?

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 - $C_1 = P \vee Q \vee R$
 - $C_2 = \neg P \vee \neg S \vee T$
 - What is a resolvent of C_1 and C_2 ? $Q \vee R \vee \neg S \vee T$

Theorem

Given two clauses C_1 and C_2 , a resolvent C of C_1 and C_2 is a logical consequence of C_1 and C_2 .

- Example: Modus ponens $(P \wedge (P \rightarrow Q) \rightarrow Q)$
 - $C_1: P$
 - $C_2: \neg P \vee Q$
 - The resolvent of C_1 and C_2 is Q which is a logical consequence of C_1 and C_2 .

Theorem

Given two clauses C_1 and C_2 , a resolvent C of C_1 and C_2 is a logical consequence of C_1 and C_2 .

Definition (Resolution principle and refutation)

Given a set S of clauses, a (resolution) deduction of C from S is a finite sequence $C_1, \dots, C_k$ of clauses such that each C_i either is a clause in S or a resolvent of clauses preceding C and $C_k = C$. A deduction of $\square$ (empty clause) is called a *refutation*.

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- If there is an argument where $P_1, \dots, P_r$ are the premises and C is the conclusion, to get a proof using the resolution principle, put $P_1, \dots, P_r$ in clause form and add to it $\neg C$ in clause form. From this sequence, if $\square$ can be derived, the argument is valid.

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- Example: *(Show that the following argument is valid. If today is Tuesday, I have a test in Mathematics or Economics. If my Economics Professor is sick, I will not have a test in Economics. Today is Tuesday, and my Economics Professor is sick. Therefore, I have a test in Mathematics.)*

$$\begin{array}{l} T \rightarrow (M \vee E) \\ S \rightarrow \neg E \\ T \wedge S \\ \hline \therefore M \end{array}$$

- What are the clauses?

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Resolution Principle

- Example:

$$\begin{array}{l} T \rightarrow (M \vee E) \\ S \rightarrow \neg E \\ T \wedge S \\ \hline \therefore M \end{array}$$

- $C_1: \neg T \vee M \vee E$
- $C_2: \neg S \vee \neg E$
- $C_3: T$
- $C_4: S$
- $C_5: \neg M$

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Resolution Principle

- Example:

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- $C_1: \neg T \vee M \vee E$
- $C_2: \neg S \vee \neg E$
- $C_3: T$
- $C_4: S$
- $C_5: \neg M$
- $C_6: \neg T \vee M \vee \neg S$

(resolvent of C_1 and C_2)

- Example:

$$\begin{array}{l} T \rightarrow (M \vee E) \\ S \rightarrow \neg E \\ T \wedge S \\ \hline \therefore M \end{array}$$

- $C_1: \neg T \vee M \vee E$

- $C_2: \neg S \vee \neg E$

- $C_3: T$

- $C_4: S$

- $C_5: \neg M$

- $C_6: \neg T \vee M \vee \neg S$

- $C_7: M \vee \neg S$

(resolvent of C_1 and C_2)

(resolvent of C_3 and C_6)

- Example:

$$\begin{array}{l} T \rightarrow (M \vee E) \\ S \rightarrow \neg E \\ T \wedge S \\ \hline \therefore M \end{array}$$

- $C_1: \neg T \vee M \vee E$

- $C_2: \neg S \vee \neg E$

- $C_3: T$

- $C_4: S$

- $C_5: \neg M$

- $C_6: \neg T \vee M \vee \neg S$

- $C_7: M \vee \neg S$

- $C_8: M$

(resolvent of C_1 and C_2)

(resolvent of C_3 and C_6)

(resolvent of C_4 and C_7)

- Example:

$$\begin{array}{l} T \rightarrow (M \vee E) \\ S \rightarrow \neg E \\ T \wedge S \\ \hline \therefore M \end{array}$$

- $C_1: \neg T \vee M \vee E$
 - $C_2: \neg S \vee \neg E$
 - $C_3: T$
 - $C_4: S$
 - $C_5: \neg M$
 - $C_6: \neg T \vee M \vee \neg S$ (resolvent of C_1 and C_2)
 - $C_7: M \vee \neg S$ (resolvent of C_3 and C_6)
 - $C_8: M$ (resolvent of C_4 and C_7)
 - $C_9: \square$ (resolvent of C_5 and C_8)
- Hence, from the resolution principle, the argument is valid.

Rules of Inference for Quantified Statements

Logic

Rules of inference for quantified statements

Rule of inference	Name
$\frac{\forall x P(x)}{\therefore ?}$	Universal instantiation
$\frac{P(c) \text{ for an arbitrary } c}{\therefore ?}$	Universal generalization
$\frac{\exists x P(x)}{\therefore ?}$	Existential instantiation
$\frac{P(c) \text{ for some element } c}{\therefore ?}$	Existential generalization
$\frac{\forall x (P(x) \rightarrow Q(x))}{P(a) \text{ where } a \text{ is a particular element in the domain}}$ $\therefore ?$	Universal modus ponens
$\frac{\forall x (P(x) \rightarrow Q(x))}{\neg Q(a) \text{ where } a \text{ is a particular element in the domain}}$ $\therefore ?$	Universal modus tollens

Table: Rules of inference for quantified statements

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Rules of inference for quantified statements

Rule of inference	Name
$\frac{\forall x P(x)}{\therefore P(c)}$	Universal instantiation
$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$	Universal generalization
$\frac{\exists x P(x)}{\therefore P(x) \text{ for some element } c}$	Existential instantiation
$\frac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$	Existential generalization
$\frac{\forall x (P(x) \rightarrow Q(x))}{P(a) \text{ where } a \text{ is a particular element in the domain}}{\therefore Q(a)}$	Universal modus ponens
$\frac{\forall x (P(x) \rightarrow Q(x))}{\neg Q(a) \text{ where } a \text{ is a particular element in the domain}}{\therefore \neg P(a)}$	Universal modus tollens

Table: Rules of inference for quantified statements

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Rules of inference for quantified statements

- Use rules of inference for quantified statements to show the premises "*A student in this class has not read the book,*" and "*Everyone in this class passed the first exam*" imply the conclusion "*Someone who has passed the first exam has not read the book.*"

Proofs

- Theorem: A mathematical statement that can be shown to be true.
 - Theorem is usually reserved for a statement that is considered at least somewhat important.
 - Less important theorems sometimes are called *propositions*.
- Axiom (or postulate): A statement that is assumed to be true.
- Lemma: A less important theorem that is helpful in the proof of other results.
- Corollary: A theorem that can be established directly from a theorem that has been proved.
- Conjecture: A statement that is being proposed to be a true statement, usually on the basis of some partial evidence.

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- Corollary: A theorem that can be established directly from a theorem that has been proved.
- Conjecture: A statement that is being proposed to be a true statement, usually on the basis of some partial evidence.
- Proof: A valid argument that establishes the truth of a Theorem.
 - The statements used in a proof can include axioms, definitions, the premises, if any, of the theorem, and previously proven theorems and uses rules of inference to draw conclusions.

- Direct proof: Used for showing statements of the form $p \rightarrow q$. We assume that p is true and use axioms, definitions, and previously proven theorems, together with rules of inference, to show that q must also be true.
- Example: Give a direct proof of the theorem “if n is an odd integer, then n^2 is odd.”

Definition (Even and odd)

The integer n is even if there exists an integer k such that $n = 2k$, and n is odd if there exists an integer k such that $n = 2k + 1$.

End