

ACOL1002: Discrete Mathematical Structures

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Logic

Predicate logic

Statement	When True	When False
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is True for every pair x, y	There is a pair x, y for which $P(x, y)$ is False
$\forall x \exists y P(x, y)$	For every x there is a y such that $P(x, y)$ is True	There is an x such that $P(x, y)$ is False for every y
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is True for every y	For every x there is a y for which $P(x, y)$ is False.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is True	$P(x, y)$ is False for every pair x, y

Table: Nested quantification of two variables.

- Determine the truth value of these statements when the domain of all variables consists of all real numbers and $Q(x, y, z)$ denotes $x + y = z$.
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- Translating mathematical statements into logical expressions involving nested quantifiers.
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 - *"Every real number except 0 has a multiplicative inverse."*

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- Translating mathematical statements into logical expressions involving nested quantifiers.
 - “*The sum of two positive integers is always positive.*”
 - $\forall x \forall y ((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$
 - “*Every real number except 0 has a multiplicative inverse.*”
 - $\forall x ((x \neq 0) \rightarrow \exists y (xy = 1))$

- Translating nested quantifiers into English.
 - Example:
 - $F(a, b)$: "*a and b are friends.*"
 - The domain consists of all students in the Institute.
 - Translate: $\exists x \forall y \forall z (F(x, y) \wedge F(x, z) \wedge (y \neq z) \rightarrow \neg(F(y, z)))$

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 - "*There is a student none of whose friends are also friends with each other.*"

- Translating English sentences into logical expressions:
- Consider following examples:
 - Translate: *"If a person is female and is a parent, then this person is someone's mother."*
 - Translate: *"Everyone has exactly one best friend."*
 - Translate: *"There does not exist a woman who has taken a flight on every airline in the world."*

Warm-up exercise

- You must be familiar with the definition of Big-O from Data Structures. Note that the definition is an interesting example of nested quantifiers.

Definition (Big-O)

Let $f(n)$ and $g(n)$ denote functions mapping positive integers to positive real numbers. The function $f(n)$ is said to be $O(g(n))$ (or $f(n) = O(g(n))$) in short if and only if there exists constants $C, n_0 > 0$ such that for all $n \geq n_0$, $f(n) \leq C \cdot g(n)$.

- How would you argue that $5n^2 + 3n + 1$ is $O(n^2)$?
- How would you argue that 2^n is **not** $O(n)$?

Rules of Inference

Logic

Rules of Inference: Propositional logic

- Consider the following argument.
 - *"If you have a current password, then you can log onto the network."*
 - *"You have a current password."*
 - Therefore *"you can log onto the network."*
- Is this a valid argument? Why?

- Argument: A sequence of statements that end with a conclusion.
- Valid argument: The conclusion must follow from the truth of the preceding statements (known as *premises*). An argument is valid if and only if it is impossible for all the premises to be true and conclusion to be false.
- Rules of inference: Templates for obtaining new statements from already available statements or in other words templates for constructing valid arguments.

Logic

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- Let us write this argument more concisely.
 - Propositions:
 - p : *"You have a current password."*
 - q : *"You can log onto the network."*
 - Argument:

$$\frac{p \rightarrow q \quad p}{\therefore q}$$

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- We know that for any propositions p, q, r , $((p \rightarrow q) \wedge p) \rightarrow q$ is a tautology.
- So, the initial argument is a valid argument.

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- We know that for any propositions p, q, r , $((p \rightarrow q) \wedge p) \rightarrow q$ is a tautology.
- So, the initial argument is a valid argument.
- Moreover, if we plug in any propositions into p and q such that the premises $p \rightarrow q$ and p are true for these propositions, then concluding q from these premises is a valid argument.
- This is called an *argument form*. The validity of an argument follows from the validity of the argument form.

Definition (Argument and argument form)

An *argument* in propositional logic is a sequence of propositions. All but the final proposition in the argument are called *premises* and the final proposition is called the *conclusion*. An argument is valid if the truth of all its premises implies that the conclusion is true.

An *argument form* in propositional logic is a sequence of compound propositions involving propositional variables. An argument form is *valid* if no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.

- How do we show that an argument form is valid?

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- How do we show that an argument form is valid?
 - Construct a truth table. However, this could be tedious.
- We first show the validity of some simple argument forms. These are called *rules of inference*. These may be used to show the validity of more complex argument forms.

Logic

Rules of inference: Propositional logic

Rule of inference	Tautology	Name
$\frac{p \quad p \rightarrow q}{\therefore ?}$	$[p \wedge (p \rightarrow q)] \rightarrow ?$	Modus ponens
$\frac{\neg q \quad p \rightarrow q}{\therefore \neg p}$	$[\neg q \wedge (p \rightarrow q)] \rightarrow ?$	Modus tollens
$\frac{p \rightarrow q \quad q \rightarrow r}{\therefore ?}$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow ?$	Hypothetical syllogism
$\frac{p \vee q \quad \neg p}{\therefore ?}$	$[(p \vee q) \wedge \neg p] \rightarrow ?$	Disjunctive syllogism

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Rules of inference: Propositional logic

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$\frac{p}{\therefore ?}$	$p \rightarrow ?$	Addition
$\frac{p \wedge q}{\therefore ?}$	$(p \wedge q) \rightarrow ?$	Simplification
$\frac{p}{q}$ $\frac{q}{\therefore ?}$	$[(p) \wedge (q)] \rightarrow ?$	Conjunction
$\frac{p \vee q}{\neg p \vee r}$ $\therefore ?$	$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow ?$	Resolution

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$\frac{p}{q}$ $\frac{q}{\therefore p \wedge q}$	$[(p) \wedge (q)] \rightarrow (p \wedge q)$	Conjunction
$\frac{p \vee q}{\neg p \vee r}$ $\frac{\neg p \vee r}{\therefore q \vee r}$	$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$	Resolution

Table: Rules of inference.

End