

ACOL1002: Discrete Mathematical Structures

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Discrete Probability

Discrete Probability

Bayes' Theorem

Theorem (Bayes' Theorem)

Suppose that E and F are events from a sample space S such that $\Pr[E] \neq 0$ and $\Pr[F] \neq 0$. Then

$$\Pr[F|E] = \frac{\Pr[E|F] \cdot \Pr[F]}{\Pr[E|F] \cdot \Pr[F] + \Pr[E|\bar{F}] \cdot \Pr[\bar{F}]}.$$

- Other Application: Bayesian spam filtering.

Discrete Probability

Bayes' Theorem

Theorem (Generalized Bayes' Theorem)

Suppose that E is an event from a sample space S and that $F_1, \dots, F_n$ are mutually exclusive events such that $\cup_{i=1}^n F_i = S$. Assume that $\Pr[E] \neq 0$ and $\Pr[F_i] \neq 0$ for $i = 1, 2, \dots, n$. Then

$$\Pr[F_j|E] = \frac{\Pr[E|F_j] \cdot \Pr[F_j]}{\sum_{i=1}^n \Pr[E|F_i] \cdot \Pr[F_i]}.$$

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Expectation and Variance

Definition (Expectation)

The *expected value*, also called the *expectation* or *mean*, of the random variable X on the sample space S is equal to

$$\mathbf{E}[X] = \sum_{s \in S} p(s) \cdot X(s).$$

The *deviation* of X at $s \in S$ is $X(s) - \mathbf{E}[X]$, the difference between the value of X and the mean of X .

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- A fair coin is flipped three times. Let S be the sample space of the eight possible outcomes, and let X be the random variable that assigns to an outcome the number of heads in this outcome. What is the expected value of X ?

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Theorem

If X is a random variable and $\mathbf{Pr}[X = r]$ is the probability that $X = r$, so that $\mathbf{Pr}[X = r] = \sum_{s \in S, X(s)=r} p(s)$, then

$$\mathbf{E}[X] = \sum_{r \in X(S)} \mathbf{Pr}[X = r] \cdot r.$$

Discrete Probability

Expectation and Variance

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- What is the expected value of the sum of the numbers that appear when a pair of fair dice is rolled?

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Expectation and Variance

Theorem

The expected number of successes when n mutually independent Bernoulli trials are performed, where p is the probability of success on each trial, is np .

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Expectation and Variance

Theorem (Linearity of expectation)

If X_i , $i = 1, 2, \dots, n$ with n a positive integer, are random variables on S , and if a and b are real numbers, then

(i) $\mathbf{E}[X_1 + X_2 + \dots + X_n] = \mathbf{E}[X_1] + \mathbf{E}[X_2] + \dots + \mathbf{E}[X_n]$,

(ii) $\mathbf{E}[aX + b] = a \cdot \mathbf{E}[X] + b$.

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Expectation and Variance

Theorem (Linearity of expectation)

If X_i , $i = 1, 2, \dots, n$ with n a positive integer, are random variables on S , and if a and b are real numbers, then

(i) $\mathbf{E}[X_1 + X_2 + \dots + X_n] = \mathbf{E}[X_1] + \mathbf{E}[X_2] + \dots + \mathbf{E}[X_n]$,

(ii) $\mathbf{E}[aX + b] = a \cdot \mathbf{E}[X] + b$.

- What is the expected value of the sum of the numbers that appear when a pair of fair dice is rolled?
- What is the expected value of the number of successes when n independent Bernoulli trials are performed, where p is the probability of success on each trial?

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Expectation and Variance

- Average-case complexity: Let the sample space I consist of all possible inputs to the algorithm. Let X be a random variable denoting the running time of the algorithm. Then the average-case complexity of the algorithm is
$$\mathbf{E}[X] = \sum_{i \in I} p(i) \cdot X(i).$$
- What is the average-case complexity of insertion sort if we just count the number of comparisons?

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Expectation and Variance

Definition (Geometric distribution)

A random variable X has a *geometric distribution with parameter p* if $\Pr[X = k] = (1 - p)^{k-1}p$ for $k = 1, 2, 3, \dots$, where p is a real number with $0 \leq p \leq 1$.

- Example: Suppose that the probability that a coin comes up tails is p . This coin is flipped repeatedly until it comes up tails. What is the expected number of flips until this coin comes up tails?

Theorem

If the random variable X has the geometric distribution with parameter p , then $\mathbf{E}[X] = 1/p$.

End