

ACOL1002: Discrete Mathematical Structures

Ragesh Jaiswal, CSE, IIT Delhi

Counting

Counting

Basic counting principles

- What are we going to count?
 - Objects with certain properties.
- Basic Counting Principles:
 - The Product Rule: Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 \cdot n_2$ ways to do the procedure.

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 - The Product Rule: Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 \cdot n_2$ ways to do the procedure.
 - Extended Product Rule: Suppose that a procedure is carried out by performing the tasks $T_1, T_2, \dots, T_m$ in sequence. If each task $T_i, i = 1, 2, \dots, n$, can be done in n_i ways, regardless of how the previous tasks were done, then there are $n_1 \cdot n_2 \dots n_m$ ways to carry out the procedure.
 - Can you argue the validity of the extended product rule using the product rule?

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- Basic Counting Principles:
 - The Product Rule: Examples
 - How many functions are there from a set of m elements to a set of n elements?
 - How many such one-one functions are there?

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- Basic Counting Principles:
 - The Product Rule:
 - The Sum Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.
 - The Extended Sum Rule: Suppose that a task can be done in one of n_1 ways, in one of n_2 ways,...,or in one of n_m ways, where none of the set of n_i ways of doing the task is the same as any of the set of n_j ways, for all pairs i and j with $1 \leq i < j \leq m$. Then the number of ways to do the task is $n_1 + n_2 + \dots + n_m$.

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- Basic Counting Principles:
 - The Product Rule:
 - The Sum Rule: Examples
 - Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Counting

Basic counting principles

- Basic Counting Principles:
 - The Product Rule:
 - The Sum Rule:
 - The Subtraction Rule: If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.
 - The subtraction rule is also known as the principle of *inclusion-exclusion*, especially when it is used to count the number of elements in the union of two sets.
 - $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$.

Counting

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- Basic Counting Principles:
 - The Product Rule:
 - The Sum Rule:
 - The Subtraction Rule:
 - The Division Rule: If f is a function from A to B where A and B are finite sets, and that for every value $y \in B$ there are exactly d values $x \in A$ such that $f(x) = y$ (in which case, we say that f is d -to-one), then $|B| = |A|/d$.
 - Example: How many different ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left neighbor and the same right neighbor?

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- Basic Counting Principles:
 - The Product Rule
 - The Sum Rule
 - The Subtraction Rule
 - The Division Rule

Counting

Tree Diagrams

- Tree diagrams: We use a branch to represent each possible choice. The possible outcomes are represented by the *leaves*.
 - Example: A playoff between two teams consists of at most five games. The first team that wins three games wins the playoff. In how many different ways can the playoff occur?

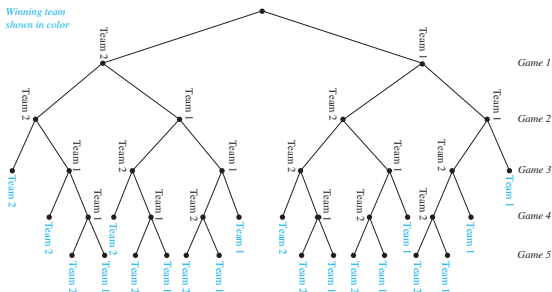


FIGURE 3 Best Three Games Out of Five Playoffs.

Counting

Pigeonhole Principle

Theorem (The Pigeonhole Principle)

If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Corollary

A function f from a set with $k + 1$ or more elements to a set with k elements is not one-to-one.

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- Show that for every integer n there is a multiple of n that has only 0s and 1s in its decimal expansion.

Counting

Pigeonhole Principle

Theorem (The Generalized Pigeonhole Principle)

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

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- What is the least number of area codes needed to guarantee that the 25 million phones in a state can be assigned distinct 10-digit telephone numbers? (Assume that telephone numbers are of the form NXX-NXX-XXXX, where the first three digits form the area code, N represents a digit from 2 to 9 inclusive, and X represents any digit.)

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- Suppose that a computer science laboratory has 15 workstations and 10 servers. A cable can be used to directly connect a workstation to a server. For each server, only one direct connection to that server can be active at any time. We want to guarantee that at any time any set of 10 or fewer workstations can simultaneously access different servers via direct connections. What is the minimum number of direct connections needed to achieve this goal?

Counting

Pigeonhole Principle: Example Applications

- During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

End