

ACOL1002: Discrete Mathematical Structures

Ragesh Jaiswal, CSE, IIT Delhi

Logic

Propositional logic: Applications

- Simplify complex sentences and enable to logically analyze them.
 - “You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old.”
 - p : “You can ride the roller coaster.”
 - q : “You are under 4 feet tall.”
 - r : “You are older than 16 years old.”
 - Express the sentence in terms of propositions p , q , and r .
 - $(q \wedge \neg r) \rightarrow \neg p$

p	q	r	$(q \wedge \neg r) \rightarrow \neg p$
T	T	T	T
T	T	F	F
T	F	T	T
F	T	T	T
T	F	F	T
F	T	F	T
F	F	T	T
F	F	F	T

- Simplify complex sentences and enable to logically analyze them.
- Translate system specification expressed in natural language into unambiguous logical expressions.
 - Example:
 - “The diagnostic message is stored in the buffer or is retransmitted.”
 - “The diagnostic message is not stored in the buffer.”
 - “If the diagnostic message is stored in the buffer, then it is retransmitted.”
 - “The diagnostic message is not retransmitted.”
 - Consistency: Whether all the specifications can be satisfied simultaneously.
 - Are these specifications consistent?

- Simplify complex sentences and enable to logically analyze them.
- Translate system specification expressed in natural language into unambiguous logical expressions.
- Resolve complex puzzling scenarios.
 - Example:
 - An island has two kinds of inhabitants, knights and knaves. Knights always tell the truth and Knaves always lie. You meet two people on this island A and B . What are A and B if A says “ B is a knight” and B says “The two of us are opposite types”?

Definition (Tautology and Contradiction)

A compound proposition that is always true, no matter what the truth values of the proposition that occurs in it, is called a tautology. A compound proposition that is always false is called a contradiction. A compound proposition that is neither a tautology nor a contradiction is called a contingency.

- Examples:
 - $(p \vee \neg p)$ is a tautology.
 - $(p \wedge \neg p)$ is a contradiction.

Definition (Logical equivalence)

A compound proposition p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

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Definition (Logical equivalence)

Compound propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

- Show that p and q are logically equivalent if and only if the columns giving their truth values match.
- Show that $\neg(p \wedge q) \equiv \neg p \vee \neg q$.

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- Show that $p \rightarrow q \equiv \neg p \vee q$.

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Propositional logic

Equivalence	Name
$p \wedge T \equiv ?$ $p \vee F \equiv ?$	Identity laws
$p \vee T \equiv ?$ $p \wedge F \equiv ?$	Domination laws
$p \vee p \equiv ?$ $p \wedge p \equiv ?$	Idempotent laws
$\neg(\neg p) \equiv ?$	Double negation law
$p \vee q \equiv ?$ $p \wedge q \equiv ?$	Commutative laws
$(p \vee q) \vee r \equiv ?$ $(p \wedge q) \wedge r \equiv ?$	Associative laws
$p \vee (q \wedge r) \equiv ?$ $p \wedge (q \vee r) \equiv ?$	Distributive laws

Table: Logical equivalences.

Logic

Propositional logic

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$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
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Propositional logic

Equivalence	Name
$\neg(p \wedge q) \equiv ?$ $\neg(p \vee q) \equiv ?$	De Morgan's laws
$p \vee (p \wedge q) \equiv ?$ $p \wedge (p \vee q) \equiv ?$	Absorption laws
$p \vee \neg p \equiv ?$ $p \wedge \neg p \equiv ?$	Negation laws
$p \rightarrow q \equiv ?$	
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$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$	Negation laws
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$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$	

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Table: Logical equivalences.

- Argue that for compound propositions p , q , and r , if $p \equiv q$ and $q \equiv r$, then $p \equiv r$.

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- Argue that for compound propositions p, q , and r , if $p \equiv q$ and $q \equiv r$, then $p \equiv r$.
- Show that $\neg(p \rightarrow q) \equiv (p \wedge \neg q)$.
- Show that $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$.
- Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

- Consider the following two statements:
 - “*All computers connected to the Institute network are functioning properly.*”
 - “*Computer-1 is connected to the Institute network.*”
- Is it ok to make the conclusion that Computer-1 is functioning properly?

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 - *"All computers connected to the Institute network are functioning properly."*
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- Is it ok to make the conclusion that Computer-1 is functioning properly?
- Can we obtain this conclusion using propositional logic?
- Suppose there are only two computers in the institute. Consider the following propositions:
 - p : Computer-1 is connected to the network.
 - q : Computer-2 is connected to the network.
 - r : Computer-1 is functioning properly.
 - s : Computer-2 is functioning properly.
- We can write $(p \rightarrow r) \wedge (q \rightarrow s) \wedge p$.

- Consider the following two statements:
 - *"All computers connected to the Institute network are functioning properly."*
 - *"Computer-1 is connected to the Institute network."*
- Is it ok to make the conclusion that Computer-1 is functioning properly?
- Can we obtain this conclusion using propositional logic?
- Suppose there are only two computers on the institute. Consider the following propositions:
 - p : Computer-1 is connected to the network.
 - q : Computer-2 is connected to the network.
 - r : Computer-1 is functioning properly.
 - s : Computer-2 is functioning properly.
- We can write $(p \rightarrow r) \wedge (q \rightarrow s) \wedge p$.
- Now, suppose there are 10,000 computers in the institute?

Predicate Logic

- Consider the following two statements:
 - “*All computers connected to the Institute network are functioning properly.*”
 - “*Computer-1 is connected to the Institute network.*”
- Is it ok to make the conclusion that Computer-1 is functioning properly?
- Suppose there are 10,000 computers in the institute?
- Consider the following concise way of writing propositions:
 - $P(x)$: x is connected to the institute network.
 - x can take values Computer-1, Computer-2 etc.
 - P denotes the *predicate* “is connected to the institute network.”
 - $P(x)$ can be thought of the value of the *propositional function* P at x .

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- Is it ok to make the conclusion that Computer-1 is functioning properly?
- Suppose there are 10,000 computers in the institute?
- Consider the following concise way of writing propositions:
 - $P(x)$: x is connected to the institute network.
 - $R(x)$: x is functioning properly.

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- Suppose there are 10,000 computers in the institute?
- Consider the following concise way of writing propositions:
 - $P(x)$: x is connected to the institute network.
 - $R(x)$: x is functioning properly.
- Are $P(x)$ and $R(x)$ propositions?

- Consider the following two statements:
 - *"All computers connected to the Institute network are functioning properly."*
 - *"Computer-1 is connected to the Institute network."*
- Is it ok to make the conclusion that Computer-1 is functioning properly?
- Suppose there are 10,000 computers on the institute network?
- Consider the following concise way of writing propositions:
 - $P(x)$: x is connected to the institute network.
 - $R(x)$: x is functioning properly.
- Are $P(x)$ and $R(x)$ propositions? No, but $P(\text{Computer-100})$ and $R(\text{Computer-200})$ are propositions.

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- Consider the following concise way of writing propositions:
 - $P(x)$: x is connected to the institute network.
 - $R(x)$: x is functioning properly.
- Are $P(x)$ and $R(x)$ propositions? No, but $P(\text{Computer-100})$ and $R(\text{Computer-200})$ are propositions.
- What we would like to say is that for any assignment of x from the set $\{\text{Computer-1}, \dots, \text{Computer-10000}\}$, $P(x) \rightarrow R(x)$.

- Quantification expresses the extent to which a predicate is true over a range of elements.
- There are two types of quantification:
 - *Universal quantification* which tells that a predicate is true for every element under consideration.
 - *Existential quantification* tells us that there is one or more element under consideration for which the predicate is true.
- The area of logic that deals with predicates and quantifiers is called *predicate calculus*.

End