

ACOL1002: Discrete Mathematical Structures

Ragesh Jaiswal, CSE, IIT Delhi

Sequences and summations

Definition (Sequence)

A sequence is a function from a subset of the set of integers (usually either the set $\{0, 1, 2, \dots\}$ or the set $\{1, 2, 3, \dots\}$) to a set S . We use the notation a_n to denote the image of the integer n . We call a_n a term of the sequence.

- We use the notation $\{a_n\}$ to describe the sequence.
- Example: $\{a_n\}$ where $a_n = 1/n$. The terms of this sequence, beginning with a_1 is $1, 1/2, 1/3, 1/4, \dots$

Definition (Geometric progression)

A geometric progression is a sequence of the form $a, ar, ar^2, \dots, ar^n, \dots$ where the initial term a and the common ratio r are real numbers.

Definition (Arithmetic progression)

An arithmetic progression is a sequence of the form $a, a + d, a + 2d, \dots, a + nd, \dots$ where the initial term a and the common difference d are real numbers.

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- Show that for any real number $r > 1$,
$$r^0 + r^1 + r^2 + \dots + r^n = \frac{r^{n+1} - 1}{r - 1}.$$
- Show that $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}.$
- Give a tight estimate on $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}.$
- Show that $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}.$

Definition (Recurrence relation)

A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \dots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer. A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

- Example: $\{a_n\}$ is a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \dots$ and $a_0 = 3$ and $a_1 = 5$.

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- The *initial conditions* for a recursively defined sequence specify the terms that precede the first term where the recurrence relation takes effect.
- We say that we have solved the recurrence relation together with the initial conditions when we find an explicit formula, called a *closed formula*, for the terms of the sequence.

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- Solve the following recurrence relation and the initial condition:
 $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \dots$ and $a_0 = 2$.

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- Solve the following recurrence relation and the initial condition:
 $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \dots$ and $a_0 = 2$. ($a_n = 3n + 2$)

Cardinality of Sets

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Definition

The sets A and B have the same cardinality if there is a one-to-one correspondence from A to B . When A and B have the same cardinality, we write $|A| = |B|$.

Definition

If there is a one-to-one function from A to B , the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$. The cardinality of A is less than the cardinality of B , written as $|A| < |B|$, if there is an injection but no surjection from A to B .

Definition (Countable and uncountable sets)

A set that is either finite or has the same cardinality as the set of positive integers is called *countable*. A set that is not countable is called *uncountable*.

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- Show that the set of odd positive integers is a countable set.

End