

ACOL1002: Discrete Mathematical Structures

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Basic Structure: Sets, Functions, Sequences, Sums, and Matrices

Basic Structures

Set operations

- Show that $\overline{A \cap B} = \bar{A} \cup \bar{B}$.
 - Show (1) $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$, and (2) $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$.
 - Use set builder notation.
 - Use a *membership table*.

Basic Structures

Set operations

Identity	Name
$A \cap U = ?$ $A \cup \emptyset = ?$	Identity laws
$A \cup U = ?$ $A \cap \emptyset = ?$	
$A \cup A = ?$ $A \cap A = ?$	Domination laws
$\overline{(\overline{A})} = ?$	
$A \cup B = B \cup ?$ $A \cap B = B \cap ?$	Idempotent laws
$\overline{(\overline{A})} = ?$	
$A \cup B = B \cup ?$ $A \cap B = B \cap ?$	Commutative laws
$A \cup (B \cup C) = ?$ $A \cap (B \cap C) = ?$	
$A \cup (B \cap C) = ?$ $A \cap (B \cup C) = ?$	Associative laws
$A \cup (B \cap C) = ?$ $A \cap (B \cup C) = ?$	
$A \cup (B \cap C) = ?$ $A \cap (B \cup C) = ?$	Distributive laws
$A \cup (B \cap C) = ?$ $A \cap (B \cup C) = ?$	

Table: Set identities.

Basic Structures

Set operations

Identity	Name
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws

Table: Set identities.

Basic Structures

Set operations

Identity	Name
$\overline{(A \cap B)} = ?$ $\overline{(A \cup B)} = ?$	De Morgan's laws
$A \cup (A \cap B) = ?$ $A \cap (A \cup B) = ?$	Absorption laws
$A \cup \overline{A} = ?$ $A \cap \overline{A} = ?$	Complement laws

Table: Set identities.

Basic Structures

Set operations

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$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
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$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
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$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$ $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Table: Set identities.

- Use set identities to show that $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$.

Basic Structures: Functions

Basic Structures

Functions

Definition (Function)

Let A and B be nonempty sets. A function f from A to B is an assignment of exactly one element of B to each element of A . We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A . If f is a function from A to B , we write $f : A \rightarrow B$.

Definition

If f is a function from A to B , we say that A is the *domain* of f and B is the *codomain* of f . If $f(a) = b$, we say that b is the *image* of a and a is a *preimage* of b . The *range*, or image, of f is the set of all images of elements of A . Also, if f is a function from A to B , we say that f maps A to B .

Basic Structures

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- Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ assign the square of an integer to this integer.
 - What is the codomain of f ?
 - What is the range of f ?

Basic Structures

Functions

Definition (real/integers-valued functions)

A function is called *real-valued* if its codomain is the set of real numbers, and it is called *integer-valued* if its codomain is the set of integers.

Definition (Sum/product of real/integer-valued functions)

Let f_1 and f_2 be functions from A to $\mathbb{R}$. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\mathbb{R}$ defined for all $x \in A$ by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x), \quad (f_1 f_2)(x) = f_1(x)f_2(x).$$

- Example: Let f_1 and f_2 be functions from $\mathbb{R}$ to $\mathbb{R}$ such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. The what are:
 - $(f_1 + f_2)(x) = ?$
 - $(f_1 f_2)(x) = ?$

Definition

Let f be a function from A to B and let S be a subset of A . The image of S under the function f is the subset of B that consists of the images of the elements of S . We denote the image of S by $f(S)$, so

$$f(S) = \{t | \exists s \in S (t = f(s))\}.$$

We also use the shorthand $\{f(s) | s \in S\}$ to denote this set.

- Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$ with $f(a) = 2, f(b) = 1, f(c) = 4, f(d) = 1,$ and $f(e) = 1$. The image of the subset $S = \{b, c, d\}$ is the set $f(S) = ?$.

Definition

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Definition (One-to-one functions)

A function f is said to be *one-to-one*, or an *injection*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f . A function is said to be *injective* if it is one-to-one.

- Consider a function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ defined as $f(x) = x^2$. Is this function one-to-one?

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- Consider a function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ defined as $f(x) = x^2$. Is this function one-to-one? **No**

Basic Structures

Functions

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Definition (Increasing/decreasing functions)

A function f whose domain and codomain are subsets of the set of real numbers is called *increasing* if $f(x) \leq f(y)$, and *strictly increasing* if $f(x) < f(y)$, whenever $x < y$ and x and y are in the domain of f . Similarly, f is called *decreasing* if $f(x) \geq f(y)$, and *strictly decreasing* if $f(x) > f(y)$, whenever $x < y$ and x and y are in the domain of f . (The word strictly in this definition indicates a strict inequality.)

- Prove or disprove: A strictly increasing function from $\mathbb{R}$ to $\mathbb{R}$ is one-to-one.

Basic Structures

Functions

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Definition (Onto functions)

A function f from A to B is called *onto*, or a *surjection*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called *surjective* if it is onto.

- Is the function $f(x) = x^2$ from $\mathbb{Z}$ to $\mathbb{Z}$ onto?

Basic Structures

Functions

Definition (One-to-one functions)

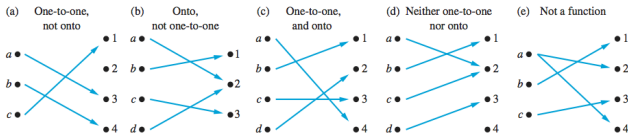
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Definition (Onto functions)

A function f from A to B is called *onto*, or a *surjection*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called *surjective* if it is onto.

Definition (Bijection)

The function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto. We also say that such a function is *bijective*.



- Suppose that $f : A \rightarrow B$.
 - To show that f is injective: Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$, then $x = y$.
 - To show that f is not injective: Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.
 - To show that f is surjective: Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.
 - To show that f is not surjective: Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Definition (Inverse function)

Let f be a one-to-one correspondence from the set A to the set B . The *inverse* function of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1} . Hence, $f^{-1}(b) = a$ when $f(a) = b$.

- Example: Let f be a function from $\mathbb{R}$ to $\mathbb{R}$ with $f(x) = x^2$. Is f invertible?

Definition (Composition of functions)

Let g be a function from the set A to the set B and let f be a function from the set B to the set C . The *composition* of the functions f and g , denoted for all $a \in A$ by $f \circ g$, is defined by $(f \circ g)(a) = f(g(a))$.

- Example: Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ and $g : \mathbb{Z} \rightarrow \mathbb{Z}$ be functions defined as $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g ? What is the composition of g and f ?
- For any function f , what is the composition of f and f^{-1} ?
- For any function f , what is the composition of f^{-1} and f ?

Basic Structures

Functions

Definition (Graph of functions)

Let f be a function from the set A to the set B . The *graph* of a function f is the set of ordered pairs $\{(a, b) | a \in A \text{ and } f(a) = b\}$.

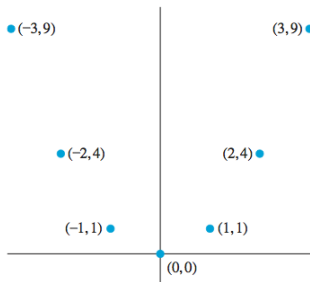


Figure: The graph of $f(x) = x^2$ from $\mathbb{Z}$ to $\mathbb{Z}$.

Basic Structures

Functions

Definition (Partial functions)

A partial function f from a set A to a set B is an assignment to each element a in a subset of A , called the domain of definition of f , of a unique element b in B . The sets A and B are called the domain and codomain of f , respectively. We say that f is undefined for elements in A that are not in the domain of definition of f . When the domain of definition of f equals A , we say that f is a total function.

- The function $f : \mathbb{Z} \rightarrow \mathbb{R}$ where $f(n) = \sqrt{n}$ is a partial function from $\mathbb{Z}$ to $\mathbb{R}$ where the domain of definition is the set of nonnegative integers.

Sequences and summations

Definition (Sequence)

A sequence is a function from a subset of the set of integers (usually either the set $\{0, 1, 2, \dots\}$ or the set $\{1, 2, 3, \dots\}$) to a set S . We use the notation a_n to denote the image of the integer n . We call a_n a term of the sequence.

- We use the notation $\{a_n\}$ to describe the sequence.
- Example: $\{a_n\}$ where $a_n = 1/n$. The terms of this sequence, beginning with a_1 is $1, 1/2, 1/3, 1/4, \dots$

Definition (Geometric progression)

A geometric progression is a sequence of the form $a, ar, ar^2, \dots, ar^n, \dots$ where the initial term a and the common ratio r are real numbers.

Definition (Arithmetic progression)

An arithmetic progression is a sequence of the form $a, a + d, a + 2d, \dots, a + nd, \dots$ where the initial term a and the common difference d are real numbers.

Basic Structures

Functions

Definition (Geometric progression)

A geometric progression is a sequence of the form $a, ar, ar^2, \dots, ar^n, \dots$ where the initial term a and the common ratio r are real numbers.

Definition (Arithmetic progression)

An arithmetic progression is a sequence of the form $a, a + d, a + 2d, \dots, a + nd, \dots$ where the initial term a and the common difference d are real numbers.

- Show that for any real number $r > 1$,
$$r^0 + r^1 + r^2 + \dots + r^n = \frac{r^{n+1} - 1}{r - 1}.$$
- Show that $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}.$
- Give a tight estimate on $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}.$

End