

ACOL1002: Discrete Mathematical Structures

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Basic Structure: Sets, Functions, Sequences, Sums, and Matrices

Basic Structures

Sets

Definition (Set)

A *set* is an unordered collection of objects, called *elements* or *members* of a set. A set is said to *contain* its elements. We write $a \in A$ to denote that a is an element of the set A . The notation $a \notin A$ denotes that a is not an element of the set A .

- Examples:

- $S_1 = \{1, 3, 5, 7, 9\}$
- $S_3 = \{x \mid x \text{ is an odd positive integer less than } 10\}$
- $S_2 = \{1, 2, 3, \dots, 99\}$
- $\mathbb{N} = \{0, 1, 2, 3, \dots\}$, the set of natural numbers.
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, the set of integers.
- $\mathbb{Z}^+ = \{1, 2, \dots\}$, the set of positive integers.
- $\mathbb{Q} = \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{Z}, \text{ and } q \neq 0\}$, the set of rational numbers.
- $\mathbb{R}$, the set of real numbers.
- $\mathbb{R}^+$, the set of positive real numbers.
- $\mathbb{C}$, the set of complex numbers.

Basic Structures

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- Examples: Intervals (closed and open)

- $[a, b] = \{x | a \leq x \leq b\}$
- $[a, b) = \{x | a \leq x < b\}$
- $(a, b] = \{x | a < x \leq b\}$
- $(a, b) = \{x | a < x < b\}$

Definition (Equality of Sets)

Two sets are *equal* if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write $A = B$ if A and B are equal sets.

- Are the following sets equal?
 - $\{1, 3, 5\}$ and $\{3, 1, 5\}$
 - $\{1, 3, 5\}$ and $\{1, 1, 3, 3, 3, 5, 5\}$
- A set with no elements is called an *empty set* or *null set*. It is denoted by $\emptyset$ or by $\{\}$.
- A set with one element is called a *singleton set*.

Basic Structures

Sets

- Venn Diagram
 - Used to represent graphically and indicate relationships between sets.
 - The *Universal set* (all objects under consideration) is represented using a rectangle.
 - Geometric figures (typically circle) inside the rectangle are used to represent sets.
 - Dots are used to represent elements.

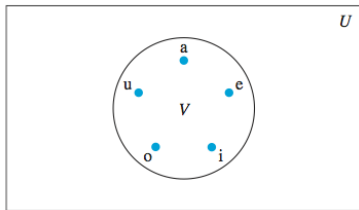


Figure: Venn diagram for the set of vowels

Basic Structures

Sets

Definition (Subset)

A set A is a *subset* of B if and only if every element of A is also an element of B . We use the notation $A \subseteq B$ to indicate that A is the subset of the set B .

- For any sets A, B , $A \subseteq B$ iff $\forall x(x \in A \rightarrow x \in B)$ is true.

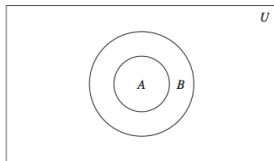


Figure: Venn diagram showing that $A \subseteq B$.

Theorem

For every set S , (i) $\emptyset \subseteq S$ and (ii) $S \subseteq S$.

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- A set A is said to be a *proper subset* of a set B if A is a subset of B but $A \neq B$.
- Write in terms of a quantified expression.

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- A set A is said to be a *proper subset* of a set B if A is a subset of B but $A \neq B$.
- Write in terms of a quantified expression:
$$\forall x(x \in A \rightarrow x \in B) \wedge \exists y(y \in B \wedge y \notin A).$$

Basic Structures

Sets

Definition (Subset)

A set A is a *subset* of B if and only if every element of A is also an element of B . We use the notation $A \subseteq B$ to indicate that A is the subset of the set B .

Theorem

Two sets A and B are equal if and only if $A \subseteq B$ and $B \subseteq A$.

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Sets

Definition (Size of a set)

Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a *finite set* and that n is the cardinality of S . The cardinality of S is denoted by $|S|$.

Definition (Infinite set)

A set is said to be infinite if it is not finite. (Example: set of positive integers)

Definition (Power set)

Given a set S , the *power set* of S is the set of all subsets of the set S . The power set of S is denoted by $\mathcal{P}(S)$.

- Examples:

- $\mathcal{P}(\{1, 2, 3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$
- $\mathcal{P}(\emptyset) = \{\emptyset\}$.
- If a set has n elements, how many elements does the power set have?

Basic Structures

Sets

Definition (Ordered n -tuple)

The ordered n -tuple $(a_1, a_2, \dots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, ..., and a_n as its n^{th} element.

Definition (Cartesian product of two sets)

Let A and B be sets. The *cartesian product* of A and B , denoted by $A \times B$, is the set of all ordered pairs (a, b) , where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}$$

- Example:

- $A = \{1, 2\}, B = \{a, b, c\}$
- $A \times B = ?$
- $B \times A = ?$

Basic Structures

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- Example:

- $A = \{1, 2\}, B = \{a, b, c\}$
- $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$
- $B \times A = \{(a, 1), (b, 1), (c, 1), (a, 2), (b, 2), (c, 2)\}$

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Definition (Cartesian product)

The Cartesian product of the sets $A_1, A_2, \dots, A_n$, denoted by $A_1 \times A_2 \times \dots \times A_n$, is the set of ordered n -tuples $(a_1, a_2, \dots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \dots, n$. In other words,

$$A_1 \times A_2 \times \dots \times A_n = \{(a_1, a_2, \dots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \dots, n\}.$$

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Definition (Relation)

A subset R of the Cartesian product $A \times B$ is called a *relation* from the set A to the set B . A relation from a set A to itself is called a relation on A .

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Sets

- Given a predicate P , and a domain D , we define the *truth set* of P to be the set of elements x in D for which $P(x)$ is true. The truth set of $P(x)$ is denoted by $\{x \in D \mid P(x)\}$.
- Examples: Consider predicates $P(x) : |x| = 1$, $Q(x) : x^2 = 2$, and $R(x) : |x| = x$ and let the domain be the set of integers.
 - Truth set of $P(x) = ?$
 - Truth set of $Q(x) = ?$
 - Truth set of $R(x) = ?$

Basic Structures

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- Examples: Consider predicates $P(x) : |x| = 1$, $Q(x) : x^2 = 2$, and $R(x) : |x| = x$ and let the domain be the set of integers.
 - Truth set of $P(x) = \{-1, 1\}$
 - Truth set of $Q(x) = \emptyset$
 - Truth set of $R(x) = \mathbb{N}$

Set operations

Basic Structures

Set operations

Definition (Union of sets)

Let A and B be sets. The *union* of the sets A and B , denoted by $A \cup B$, is the set that contains those elements that are in A or in B (this includes the element being present in both).

- $A \cup B = \{x | x \in A \vee x \in B\}.$

Definition (Intersection of sets)

Let A and B be sets. The *intersection* of the sets A and B , denoted by $A \cap B$, is the set that contains those elements that are both in A and in B .

- $A \cap B = \{x | x \in A \wedge x \in B\}.$
- Two sets are called *disjoint* if their intersection is the empty set.
- Show that $|A \cup B| = |A| + |B| - |A \cap B|.$

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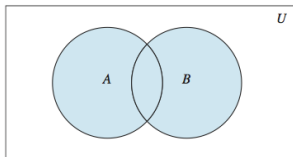
Set operations

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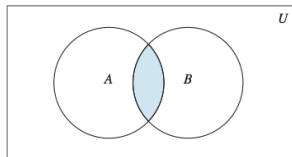
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$A \cup B$ is shaded.



$A \cap B$ is shaded.

Basic Structures

Set operations

Definition (Difference of sets)

Let A and B be sets. The *difference* of the sets A and B , denoted by $A - B$, is the set containing those elements that are in A but not in B . The difference of A and B is also called the *complement* of B with respect to A .

- $A - B = \{x | x \in A \wedge x \notin B\}$.
- The difference of sets A and B is sometimes denoted by $A \setminus B$.

Definition (Complement of a set)

Let U be the universal set. The *complement* of the set A denoted by $\bar{A}$ is the complement of A with respect to U . Therefore, the complement of the set A is $U - A$.

- $\bar{A} = \{x \in U | x \notin A\}$.
- Show that $A - B = A \cap \bar{B}$.

Basic Structures

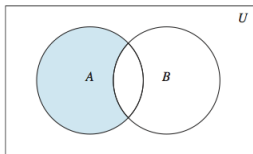
Set operations

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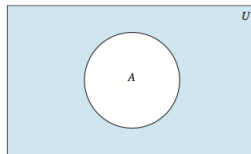
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End