

Name: _____

Entry number: _____

There are 2 questions for a total of 10 points.

1. (1 $\frac{1}{2}$ points) State True or False: Any simple, undirected graph with $n > 1$ nodes and $(n - 1)$ edges has a degree-1 node.

(Remember, a simple undirected graph is one that does not have any self-loops or multi-edges)

1. ~~True~~ False

Solution:

The argument below holds for connected graphs. After the initial round of grading, a student pointed out that the question did not explicitly state that the graph is connected. Given this, there is a simple counterexample of a graph with 4 nodes where three are connected in a triangle, and one node is isolated. So, the answer is False.

(This is just for explanation of the answer. You were not asked to give a justification).

We will prove by contradiction.

Assume for the sake of contradiction that there is a simple, **connected**, undirected graph G with $n > 1$ nodes and $n - 1$ edges that does not have a degree-1 node. Since every node has degree ≥ 2 , we get that the sum of degrees of nodes in the graph is at least $2n$. We also know that the sum of the degrees of nodes in the graph equals twice the number of edges. So, we get that:

$$2 \cdot \text{number of edges} = \text{sum of degrees} \geq 2n.$$

So, the number of edges is $\geq n$, which contradicts the fact that the number of edges is $n - 1$. This proves the given statement.

2. Answer the following questions:

- (a) (1 $\frac{1}{2}$ points) State True or False: For any positive integer n , if n leaves a remainder 2 when divided by 4, then n cannot be written as the difference of squares of two integers.

(a) True

- (b) (7 points) Give a justification for your answer to part (a).

Solution: We will prove the statement by contradiction.

For the sake of contradiction, assume that there is a positive integer n that leaves a remainder of 2 when divided by 4 and can be written as a difference of squares of two positive integers p and q , i.e., $n = p^2 - q^2$. We obtain a contradiction for each of these cases for p and q :

- Case 1: (p and q are even)

Let $p = 2k_1$ and $q = 2k_2$ for integers k_1, k_2 . Then $p^2 - q^2 = 4 \cdot (k_1^2 - k_2^2)$, which leaves a remainder 0 when divided by 4.

- Case 2: (p is odd and q is even)
Let $p = 2k_1 + 1$ and $q = 2k_2$ for integers k_1, k_2 . Then $p^2 - q^2 = 4 \cdot (k_1^2 + k_1 - k_2^2) + 1$, which leaves a remainder 1 when divided by 4.
- Case 3: (p is even and q is odd)
Let $p = 2k_1$ and $q = 2k_2 + 1$ for integers k_1, k_2 . Then $p^2 - q^2 = 4 \cdot (k_1^2 - k_2^2 - k_2) - 1$, which leaves a remainder 3 when divided by 4.
- Case 4: (p is odd and q is odd)
Let $p = 2k_1 + 1$ and $q = 2k_2 + 1$ for integers k_1, k_2 . Then $p^2 - q^2 = 4 \cdot (k_1^2 + k_1 - k_2^2 - k_2)$, which leaves a remainder 0 when divided by 4.

Since in none of the cases the difference of squares leaves a remainder 2 when divided by 4, we get that n does not leave a remainder 2 when divided by 4. This contradicts our initial assumption. So, we conclude that for any positive integer n , if n leaves a remainder 2 when divided by 4, then n cannot be written as the difference of squares of two integers.