

Name: \_\_\_\_\_

Entry number: \_\_\_\_\_

There are 2 questions for a total of 10 points.

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1. (4 points) Prove or disprove: *The sum of any three consecutive positive integers is divisible by 3.*

**Solution:** We will prove the statement. Consider an arbitrary sequence of three consecutive positive integers  $a, a + 1, a + 2$ . We consider the following exhaustive cases:

- Case 1:  *$a$  leaves a remainder of 0 when divided by 3*

In this case, we can write  $a = 3k$  for some positive integer  $k$ . So,  $a + 1 = 3k + 1$  and  $a + 2 = 3k + 2$ . So, we get  $a + (a + 1) + (a + 2) = 9k + 3$ , which is divisible by 3.

- Case 2:  *$a$  leaves a remainder of 1 when divided by 3*

In this case, we can write  $a = 3k + 1$  for some positive integer  $k$ . So,  $a + 1 = 3k + 2$  and  $a + 2 = 3k + 3$ . So, we get  $a + (a + 1) + (a + 2) = 9k + 6$ , which is divisible by 3.

- Case 3:  *$a$  leaves a remainder of 2 when divided by 3*

In this case, we can write  $a = 3k + 2$  for some positive integer  $k$ . So,  $a + 1 = 3k + 3$  and  $a + 2 = 3k + 4$ . So, we get  $a + (a + 1) + (a + 2) = 9k + 9$ , which is divisible by 3.

Since in all the above three cases the sum of the numbers is divisible by 3, we conclude that the sum of any sequence of three positive integers is divisible by 3.

2. Answer the following questions on Predicate Logic.

(a) (3 points) Consider the following predicates:

|                                  |                              |
|----------------------------------|------------------------------|
| $D(y)$ : $y$ is a dog.           | $B(y)$ : $y$ barks at night. |
| $P(y)$ : $y$ is a pet.           | $H(x, y)$ : $x$ has $y$ .    |
| $L(x)$ : $x$ is a light sleeper. |                              |

Express each of the statements below using quantifiers and the predicates given above. The domain of the variables  $x, y$  is the set of all living creatures.

|       | Statement                                                          | Quantified expression                                             |
|-------|--------------------------------------------------------------------|-------------------------------------------------------------------|
| $S_1$ | All dogs bark at night.                                            | $\forall y(D(y) \rightarrow B(y))$                                |
| $S_2$ | Light sleepers do not have any living creatures that bark at night | $\forall x(L(x) \rightarrow \neg \exists y(H(x, y) \wedge B(y)))$ |
| $S_3$ | All pets are dogs.                                                 | $\forall y(P(y) \rightarrow D(y))$                                |
| $S_4$ | If John is a light sleeper, then John does not have any pets.      | $L(John) \rightarrow \neg \exists y(H(x, y) \wedge P(y))$         |

(b) (3 points) Using the expressions  $S_1, S_2, S_3$ , and  $S_4$  obtained above, determine whether the following argument form is valid. Justify your answer using rules of inference.

$$\begin{array}{l} S_1 \\ S_2 \\ S_3 \\ \hline \therefore S_4 \end{array}$$

**Solution:** We obtain the following argument form by replacing the  $S_1, \dots, S_5$  above.

$$\begin{array}{l} \forall y [D(y) \rightarrow B(y)] \\ \forall x(L(x) \rightarrow \neg \exists y(H(x, y) \wedge B(y))) \\ \forall y(P(y) \rightarrow D(y)) \\ \hline \therefore L(John) \rightarrow \neg \exists y(H(x, y) \wedge P(y)) \end{array}$$

We will show that the above argument form is valid using the rules of inference:

1.  $\forall x(L(x) \rightarrow \neg \exists y(H(x, y) \wedge B(y)))$  (Premise)
2.  $L(John) \rightarrow \neg \exists y(H(John, y) \wedge B(y))$  (Universal instantiation from (1))
3.  $\forall y(\neg L(John) \vee \neg H(John, y) \vee \neg B(y))$  (De Morgan's law using (2) and  $p \rightarrow q \equiv (\neg p \vee q)$ )
4.  $\neg L(John) \vee \neg H(John, c) \vee \neg B(c)$  for an arbitrary  $c$  (Universal Instantiation from (3))
5.  $\forall y [D(y) \rightarrow B(y)]$  (Premise)
6.  $D(c) \rightarrow B(c)$  (Universal Instantiation from (5))
7.  $\forall y(P(y) \rightarrow D(y))$  (Premise)
8.  $P(c) \rightarrow D(c)$  (Universal Instantiation from (7))

9.  $P(c) \rightarrow B(c)$  (Hypothetical Syllogism on (6) and (8))

10.  $\neg P(c) \vee B(c)$  (From (9) using  $p \rightarrow q \equiv (\neg p \vee q)$ )

11.  $\neg L(John) \vee \neg H(John, c) \vee \neg P(c)$  (Resolvent of (4) and (10))

12.  $\forall y(\neg L(John) \vee \neg H(John, y) \vee \neg P(y))$  (Universal generalization using (11))

13.  $L(John) \rightarrow \neg \exists y(H(John, y) \wedge P(y))$  (De Morgan's law using (12) and  $p \rightarrow q \equiv (\neg p \vee q)$ )