There are 4 questions for a total of 100 points.

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1. (20 points) Given an $n \times n$ matrix A, use a known SVD algorithm (*given* A, *it outputs* $n \times n$ *matrices* U, V, D *such that* $A = UDV^T$) to find a matrix B such that $B^T B = I$ and the cost function $||A - B||_F$ is minimised. Discuss the correctness of your algorithm.

2. (20 points) Suppose we flip the quantifiers in the statement of the JL theorem.

   (JL Flipped) For any $0 < \varepsilon < 1$ and any integer n, let $k \geq \frac{3}{c\varepsilon^2} \ln n$ with c as in the Random Projection Theorem. The random projection $f : \mathbb{R}^d \to \mathbb{R}^k$ defined before has the property that for any set of n points in $\mathbb{R}^d$, for all pairs of points $\mathbf{v_i}$ and $\mathbf{v_j}$, with probability at least $(1 - \frac{3}{2n})$,
   $$(1 - \varepsilon)\sqrt{k}||\mathbf{v_i} - \mathbf{v_j}|| \leq ||f(\mathbf{v_i}) - f(\mathbf{v_j})|| \leq (1 + \varepsilon)\sqrt{k}||\mathbf{v_i} - \mathbf{v_j}||.$$

   Is the above statement true? Given reasons.

3. (30 points) The k-median problem is defined as follows: Given a set $X \subset \mathbb{R}^d$ in d-dimensional Euclidean space, find a set $C \subset \mathbb{R}^d$ of k points (*called centers*) such that the following cost function is minimised:
   $$cost(X, C) \equiv \sum_{x \in X} \min_{c \in C} ||x - c||.$$

   Suppose there is an *approximation scheme* for the problem, which means that there is an algorithm A that takes as input the instance (X, k) and an error parameter $\varepsilon > 0$ and outputs a $(1 + \varepsilon)$-approximate solution. Suppose the running time of the algorithm A is $O(nk2^d)$, where $n = |X|$. Can you improve the dependency of the running time on the dimension using the JL theorem? In other words, you are asked to design a faster approximation scheme. Discuss.

4. (30 points) Consider the hypothesis class of *Linear Threshold Functions (LTFs)* in $\mathbb{R}^d$. An LTF is defined using vectors $\mathbf{w}, \mathbf{b} \in \mathbb{R}^d$ with the label function being:
   $$f(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{w}^T \cdot \mathbf{x} + \mathbf{b} > 0 \\ 0, & \text{otherwise} \end{cases}$$

   Show that the VC dimension of this concept class is $d + 1$.