# COL351: Analysis and Design of Algorithms

Ragesh Jaiswal, CSE, IITD

- Main Idea: Reduction
  1. We will obtain an algorithm $A$ for a *Network Flow* problem.
     - We discussed Ford-Fulkerson and there are many more.
  2. Given a new problem, we will *rephrase* this problem as a Network Flow problem.
     - Given a bipartite graph, we constructed a network graph by adding edges from a source to vertices in the left and edges from vertices on the right to a sink etc.
  3. We will then use algorithm $A$ to solve the rephrased problem and obtain a solution.
     - We used Ford-Fulkerson to obtain a flow with maximum value for the network graph constructed.
  4. Finally, we build a solution for the original problem using the solution to the rephrased problem.
     - We used the flow to construct a matching and then argued that this will be a maximum matching.

**Definition (Perfect Matching)**

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### Theorem (Hall's Theorem)

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- Claim 3: If there is a perfect matching, then for all subsets $A \subseteq X$, $|A| \leq |N(A)|$.
- Claim 4: If there is no perfect matching, then there is a subset $A \subseteq X$ such that $|A| > |N(A)|$.

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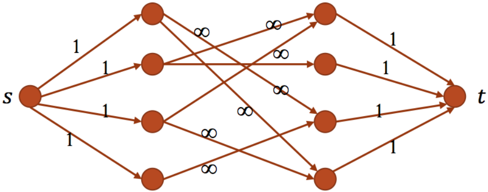
- Consider the flow network in figure below constructed using the bipartite graph.
- Claim 4.1: The max-flow in the network is equal to the maximum matching in $G$.



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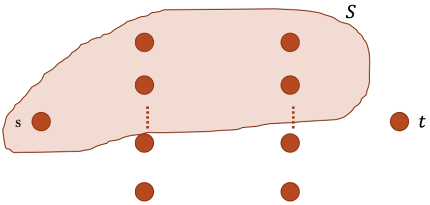
- Let $f$ be the max integer flow in the network. Consider the residual graph $G_f$. Let $S$ be the set of vertices reachable from $s$ in $G_f$. Let $A'$ be vertices of $X$ in $S$ and $B'$ be vertices of $Y$ in $S$.



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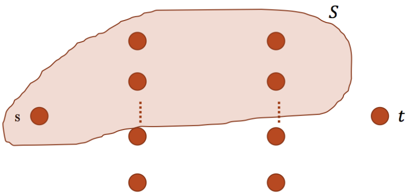

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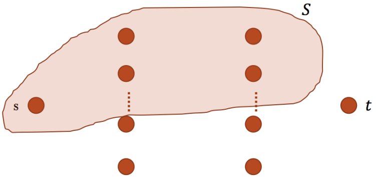
- Claim 4.2: $B' = N(A')$.
- Capacity of the cut $(S, T) = n - |A'| + |N(A')|$.



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- From Max-flow-min-cut argument (Slide 4 of this lecture), we have:
  $n - |A'| + |N(A')| = \text{max flow} < n \Rightarrow |A'| > |N(A')|$.

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- This is a constructive proof since we can find a subset $A'$ such that $|A'| > |N(A')|$.
- Such an $A'$ may be interpreted as a *certificate* of the fact that there is no perfect matching in $G$.

- Suppose there are four teams in IPL with their current number of wins:
  - Daredevils: 10
  - Sunrisers: 10
  - Lions: 10
  - Supergiants: 8
- There are 7 more games to be played. These are as follows:
  - Supergiants plays all other 3 teams.
  - Daredevils Vs Sunrisers, Sunrisers Vs Lions, Daredevils Vs Lions, Sunrisers Vs Daredevils

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- A team is said to be eliminated if it cannot end with maximum number of wins.
- Can we say that Supergiants have been eliminated give the current scenario?

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  - 4 games between Daredevils and Sunrisers.
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## Problem

There are $n$ teams. Each team $i$ has a current number of wins denoted by $w(i)$.There are $G(i,j)$ games yet to be played between team $i$ and $j$. Design an algorithm to determine whether a given team $x$ has been eliminated.

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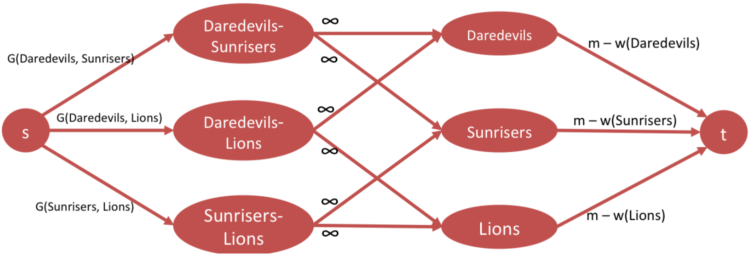


Figure: Team $x$ can end with at most $m$ wins, i.e., $m = w(x) + \sum_j G(x,j)$

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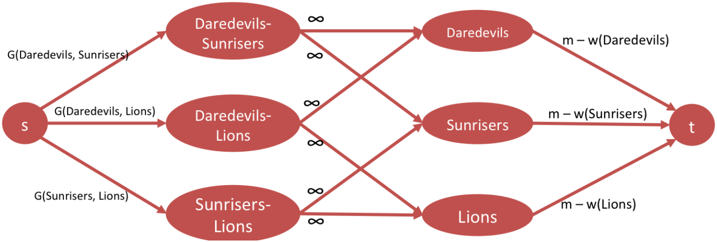


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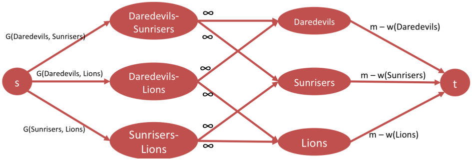

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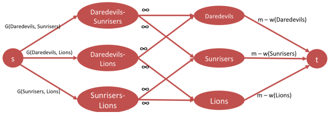

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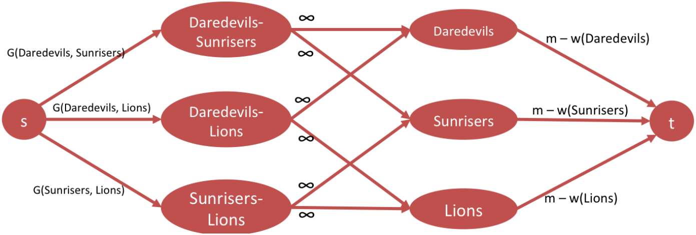

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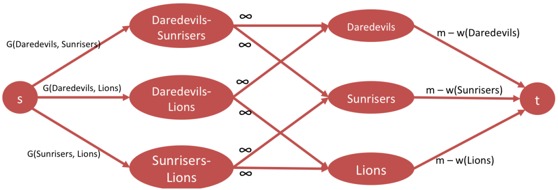

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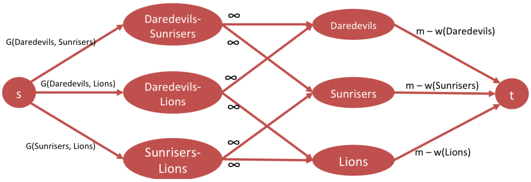

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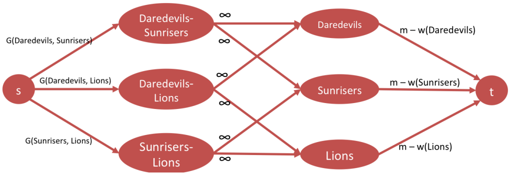

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- Let $T$ be the set of teams such that $i \in T$ **iff** $v_i \in A$. Then we have:

$$C(A,B) = \sum_{i \in T}(m - w(i)) + \sum_{\{i,j\} \not\subset T} G(i,j) < g^*$$

$$\Rightarrow \quad m \cdot |T| - \sum_{i \in T} w(i) + (g^* - \sum_{\{i,j\} \subset T} G(i,j)) < g^*$$

$$\Rightarrow \quad \sum_{i \in T} w(i) + \sum_{\{i,j\} \subset T} G(i,j) > m \cdot |T| \quad \square$$

$\square$

End