Lecture 1: Analyzing algorithms

A royal mathematical challenge (1202):

Suppose that rabbits take exactly one month to become fertile, after which they produce one child per month, forever. Starting with one rabbit, how many are there after n months?

Leonardo da Pisa, aka Fibonacci

The proliferation of rabbits

Rabbits take one month to become fertile, after which they produce one child per month, forever.

	Fertile	Not fertile
Initially		
One month		
Two months		
Three months		
Four months		
Five months		

The Fibonacci sequence

$F_1 = 1$, $F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$

These numbers grow *very* fast: $F_{30} > 10^6$!

In fact, $F_n \approx 2^{0.694n} \approx 1.6^n$, exponential growth.

The Fibonacci sequence

$F_1 = 1$, $F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$

Can you see why $F_n < 2^n$?

Computing Fibonacci numbers

```
function Fib1(n)
if n = 1 return 1
if n = 2 return 1
return Fib1(n-1) + Fib1(n-2)
```

A *recursive* algorithm

Two questions we always ask about algorithms:

Does it work correctly?

How long does it take?

Running time analysis

```
function Fib1(n)
if n = 1 return 1
if n = 2 return 1
return Fib1(n-1) + Fib1(n-2)
```

Exponential time... how bad is this?

Eg. Computing F_{200} needs about 2^{140} operations.
How long does this take on a fast computer?

IBM Summit

Can perform up to 200 quadrillion (= 200×10^{15}) operations per second.

Is exponential time all that bad?

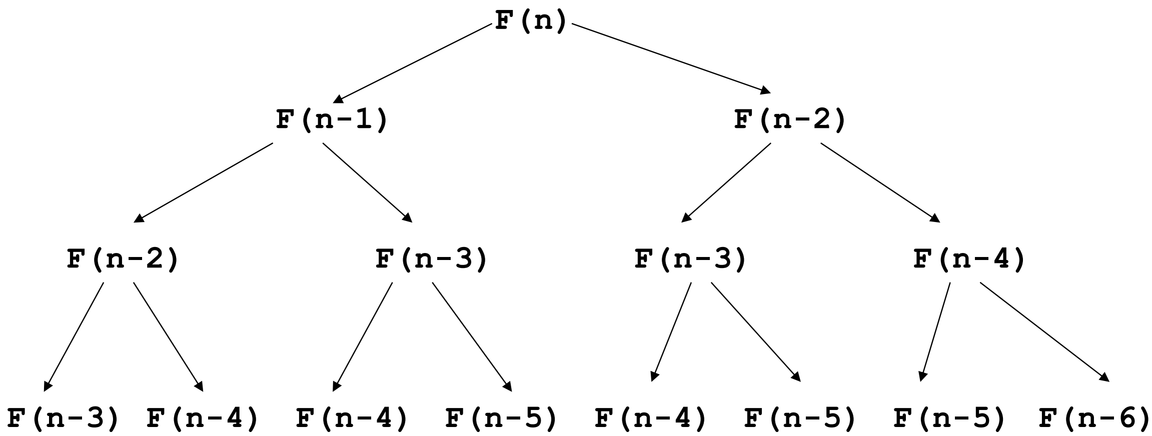
The Summit needs 2^{82} seconds for F_{200}.

Time in seconds	Interpretation
2^{10}	17 minutes
2^{20}	12 days
2^{30}	32 years
2^{40}	cave paintings
2^{45}	*homo erectus* discovers fire
2^{51}	extinction of dinosaurs
2^{57}	creation of Earth
2^{60}	origin of universe

Post mortem

What takes so long?
Let's unravel the recursion…

```
function Fib1(n)
if n = 1 return 1
if n = 2 return 1
return Fib1(n-1) + Fib1(n-2)
```



The same subproblems get solved over and over again!

A better algorithm

Subproblems: $F_1, F_2, \ldots, F_n$. Solve them in order and *save their values*!

```
function Fib2(n)
Create an array fib[1..n]
fib[1] = 1
fib[2] = 1
for i = 3 to n:
    fib[i] = fib[i-1] + fib[i-2]
return fib[n]
```

[1] Does it return the correct answer?

[2] How fast is it?

Polynomial vs. exponential

Polynomial running times:

Exponential running times:

To an excellent first approximation:

polynomial is reasonable

exponential is not reasonable

This is one of the most fundamental dichotomies in the analysis of algorithms.

A more careful analysis

```
function Fib1(n)
if n = 1 return 1
if n = 2 return 1
return Fib1(n-1) + Fib1(n-2)
```

```
function Fib2(n)
Create an array fib[1..n]
fib[1] = 1
fib[2] = 1
for i = 3 to n:
    fib[i] = fib[i-1] + fib[i-2]
return fib[n]
```

Problem: we cannot count these additions as single operations!

How many bits does F_n have?

Addition of *n*-bit numbers takes $O(n)$ time.

Fib1: $O(n\ 2^{0.7n})$ time

Fib2: $O(n^2)$ time

Addition

Adding two *n*-bit numbers takes *O(n)* simple operations:

E.g. 22 + 13:

[22]	1	0	1	1	0
[13]		1	1	0	1

Big-O notation

```
function Fib2(n)
Create an array fib[1..n]
fib[1] = 1
fib[2] = 1
for i = 3 to n:
    fib[i] = fib[i-1] + fib[i-2]
return fib[n]
```

Running time is proportional to n^2.

But what is the constant: is it $2n^2$ or $3n^2$ or what?

The constant depends upon:

The units of time – minutes, seconds, milliseconds,…

Specifics of the computer architecture.

It is *much* too hairy to figure out exactly. Moreover it is nowhere as important as the huge gulf between n^2 and 2^n.

So we simply say the running time is $O(n^2)$.