COL863: Quantum Computation and Information

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Quantum Mechanics

- Claim: (Projective measurement + unitary operators) = generalised measurement.

Proof sketch

- Let Q be the state space of the quantum system in which we would like to make a generalised measurement using measurement operators M_m.
- We introduce an ancilla system with state space M with orthonormal basis $|m\rangle$.
- Let U be an operator defined as

$$U|\psi\rangle|0\rangle \equiv \sum_m M_m|\psi\rangle|m\rangle$$

  where $|0\rangle$ is an arbitrary state of M.
- Claim 1: U preserves inner products between states of the form $|\psi\rangle|0\rangle$.

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- Claim 1: U preserves inner products between states of the form $|\psi\rangle|0\rangle$.
- Claim 2: U can be extended to a unitary operator on $Q \otimes M$ (let us denote this by U itself).
- Claim 3: Let $P_m = I_Q \otimes |m\rangle\langle m|$. Projective measurement using P_m on $Q \otimes M$ is similar to generalised measurement using M_m on Q.

Superdense coding problem

Alice wants to send two classical bits to Bob. They share a Bell pair and the constraint is that Alice can only send a single qubit to Bob.

- The trace of a square matrix is defined to be the sum of diagonal elements. That is:

$$tr(A) \equiv \sum_i A_{ii}$$

  - Exercise: Show that $tr(AB) = tr(BA)$.
  - Exercise: Show that $tr(A + B) = tr(A) + tr(B)$.
  - Exercise: Show that $tr(zA) = ztr(A)$.
  - Exercise: Show that the trace operator is invariant under change of basis.
  - Exercise: Show that for any orthonormal basis $|i\rangle$,

$$tr(A) = \sum_i \langle i| A |i\rangle .$$

  - Exercise: Show that for any unit vector $|\psi\rangle$,

$$tr(A |\psi\rangle \langle\psi|) = \langle\psi| A |\psi\rangle .$$

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  - Why: Talking about individual subsystems of a composite system becomes simpler.

- We formulated the postulates of Quantum Mechanics using state vectors.
- An alternative and mathematically equivalent formulation is through density operators and density matrices.
  - Why: Talking about individual subsystems of a composite system becomes simpler.
- The density operator is used to describe an ensemble of pure states $\{p_i, |\psi_i\rangle\}$. That is, a quantum system that is in state $|\psi_i\rangle$ with probability p_i.
  - Question Can you give a scenario where it may be useful to describe such an ensemble of states?
- Density operator: The density operator of such a system is defined by:

$$\rho \equiv \sum_i p_i |\psi_i\rangle \langle\psi_i|$$

- The density operator is often known as density matrix.

End