COL863: Quantum Computation and Information

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Introduction

- What is a qubit? Quantum analogue of classical bit.
- Classical bit can be realised in real physical systems. Does it hold for qubits? We will work with yes.
- The classical bit has two states 0 and 1. Is qubit similar?
  - Summary: The state of a qubit is a *unit* vector in a two-dimensional complex vector space with $|0\rangle$ and $|1\rangle$ as the orthonormal basis (interpreted as computational basis states).
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  - A two qubit system can be written as a superposition of computational basis states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$:

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  - Yes. The Quantum counterpart of classical circuits are called quantum circuits that has quantum gates.

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Quantum Circuit

- Single qubit gates:
  There is only one single-input logical gate in the classical setting, the `NOT` gate. What could be a quantum version of such a gate?
  - The general state of a qubit is expressed as $\alpha|0\rangle + \beta|1\rangle$. The quantum version of `NOT` gate does the following conversion:

    $$\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|1\rangle + \beta|0\rangle$$

    This is known as the X gate.
  - The general state of a qubit can be written using matrix notation as $\left[\begin{smallmatrix}\alpha \\ \beta\end{smallmatrix}\right]$. The X gate operating on the qubit can then be interpreted as a simple matrix multiplication where $X \equiv \left[\begin{smallmatrix}0 & 1 \\ 1 & 0\end{smallmatrix}\right]$.
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    - Is $\left[\begin{smallmatrix}1 & 1 \\ 1 & 0\end{smallmatrix}\right]$ a valid single-qubit gate? No
    - In general, if the state after applying the gate is $\alpha' |0\rangle + \beta' |1\rangle$, then $|\alpha'|^2 + |\beta'|^2 = 1$.
    - A necessary condition to ensure this is that the matrix is unitary. That is, $U^{\dagger} U = I$.
    - This also happens to be a sufficient condition for any quantum gate.
    - One implication of this fact is that there can be infinitely many single-qubit gates.

- Single qubit gates: Frequently used gates
  - X gate: Analogue of classical `NOT` gate with matrix representation

$$X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

  - Z gate: Matrix representation:

$$Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

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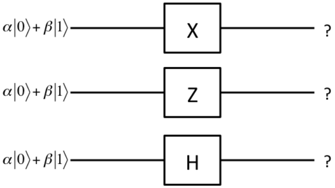

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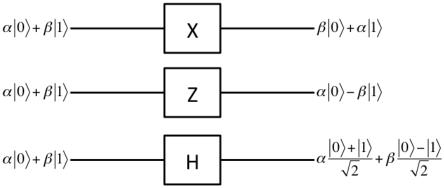

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  - Does a quantum analogue of `NAND` gate exist? No
    - `NAND` gate is irreversible. That is one cannot obtain A and B from $A \wedge B$.
    - Quantum gates are constrained to be reversible.
    - Unitary gates (operations using unitary matrices) are invertible and hence reversible.

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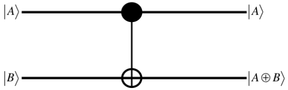

    - More precisely, the matrix representing the gate is given by

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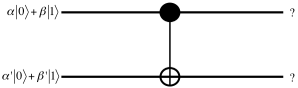

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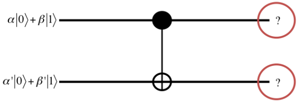

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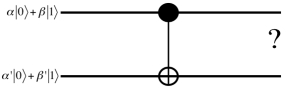

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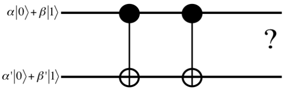

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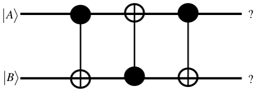

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    - Claim: *Any multiple qubit logic gate may be composed from* CNOT *and single qubit gates.*

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    - We can measure in any orthonormal basis $|a\rangle, |b\rangle$. If the state of the qubit can be expressed as $\alpha |a\rangle + \beta |b\rangle$, then the measurement result is a with probability $|\alpha|^2$ and b with probability $|\beta|^2$.
    - One such popular basis is the $|+\rangle, |-\rangle$ basis that are expressed as $|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$ and $|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$.
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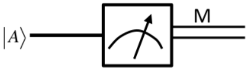
    - In quantum circuit diagrams, measurement of a qubit is represented as below:



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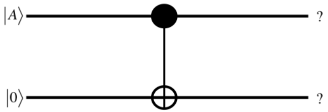

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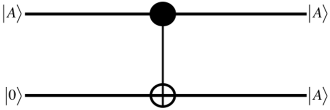

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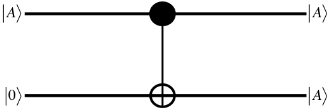

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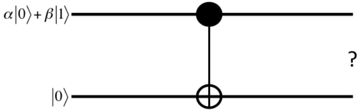

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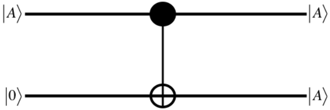

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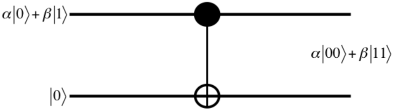

End