Name: ______________________________

Entry number: ________________________

There are 3 questions for a total of 10 points.

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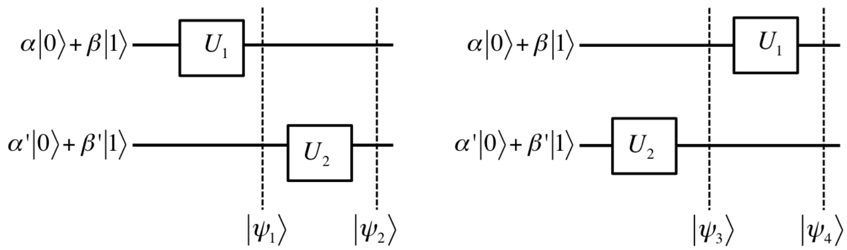
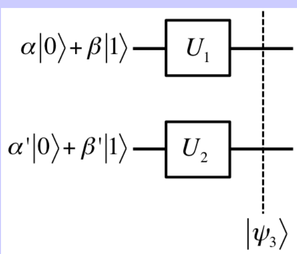
1. (3 points) Let the matrix representation of gates U_1 and U_2 be $U_1 = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ and $U_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Give the states $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle, |\psi_4\rangle$ in the circuits below.

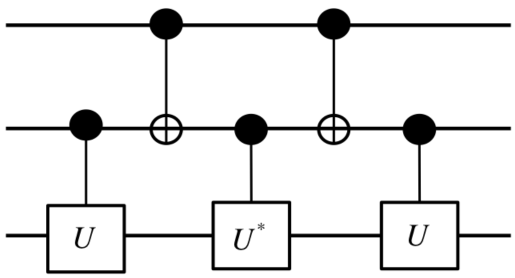


Solution: Here are the states:
- $|\psi_1\rangle = [(p\alpha + q\beta)\,|0\rangle + (r\alpha + s\beta)\,|1\rangle][\alpha'\,|0\rangle + \beta'\,|1\rangle]$
- $|\psi_2\rangle = [(p\alpha + q\beta)\,|0\rangle + (r\alpha + s\beta)\,|1\rangle][(a\alpha' + b\beta')\,|0\rangle + (c\alpha' + d\beta')\,|1\rangle]$
- $|\psi_3\rangle = [\alpha\,|0\rangle + \beta\,|1\rangle][(a\alpha' + b\beta')\,|0\rangle + (c\alpha' + d\beta')\,|1\rangle]$
- $|\psi_4\rangle = |\psi_2\rangle$

The goal of this exercise was just to make you comfortable with calculations. The main takeaway point of this simple exercise was to realise that the order in which the unitary gates are applied is not important as far as the final output state is concerned. So, this circuit may have been drawn as given below. This is what is done in the third problem.



2. (2 points) What is the input-output behaviour of the following circuit. (U^* *denotes conjugate transpose.*)

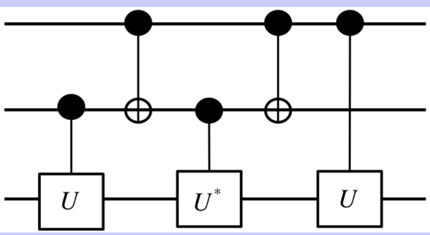


Input	Output
$\|00\rangle\|\psi\rangle$	
$\|01\rangle\|\psi\rangle$	
$\|10\rangle\|\psi\rangle$	
$\|11\rangle\|\psi\rangle$	

Solution: Here the input/output table:

Input	Output
$\|00\rangle\,\|\psi\rangle$	$\|00\rangle\,\|\psi\rangle$
$\|01\rangle\,\|\psi\rangle$	$\|01\rangle\, U(\|\psi\rangle)$
$\|10\rangle\,\|\psi\rangle$	$\|10\rangle\, U^*(\|\psi\rangle)$
$\|11\rangle\,\|\psi\rangle$	$\|11\rangle\, U^2(\|\psi\rangle)$

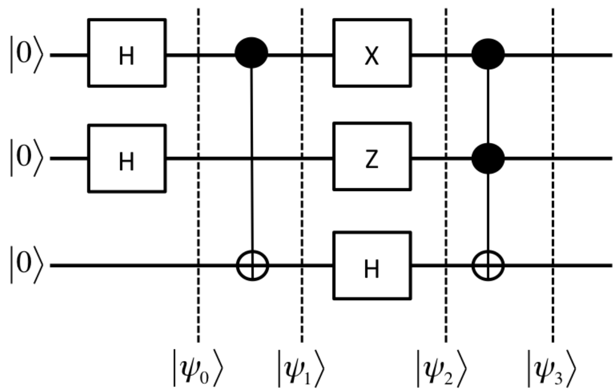
The following related circuit has a very interesting input/output behaviour (*actually, I intended to give this circuit in the problem but made a drawing error*):



Input	Output
$\|00\rangle\|\psi\rangle$	$\|00\rangle\|\psi\rangle$
$\|01\rangle\|\psi\rangle$	$\|01\rangle\|\psi\rangle$
$\|10\rangle\|\psi\rangle$	$\|10\rangle\|\psi\rangle$
$\|11\rangle\|\psi\rangle$	$\|11\rangle U^2(\|\psi\rangle)$

Note that the behaviour of this circuit is same as the Toffoli gate in case $U^2 = X$. That is, U is the "square-root" of the X gate. Much of the linear algebra theory that we will be discussing in the next few classes will be to understand why such a square-root gate will exist for *any* unitary gate (not just X gate). (*People who attended Umesh Vazirani's talk may remember him briefly talking about the ability to finely divide unitary gates into components.*) The other takeaway message of this exercise is that Toffoli gate can be built using `CNOT` and controlled-U gates.

3. (5 points) Give the the intermediate states $|\psi_0\rangle, |\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle$ of the 3-qubit circuit given below. Show your calculations.



Solution: Here are the states:

$$|\psi_0\rangle = \frac{1}{2}|000\rangle + \frac{1}{2}|010\rangle + \frac{1}{2}|100\rangle + \frac{1}{2}|110\rangle$$

$$|\psi_1\rangle = \frac{1}{2}|000\rangle + \frac{1}{2}|010\rangle + \frac{1}{2}|101\rangle + \frac{1}{2}|111\rangle$$

$$\begin{aligned} |\psi_2\rangle &= \frac{1}{2}[|1\rangle][|0\rangle]\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right] + \frac{1}{2}[|1\rangle][-|1\rangle]\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right] + \frac{1}{2}[|0\rangle][|0\rangle]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] + \frac{1}{2}[|0\rangle][-|1\rangle]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \\ &= \frac{(|000\rangle - |001\rangle - |010\rangle + |100\rangle + |011\rangle - |110\rangle + |101\rangle - |111\rangle)}{2\sqrt{2}} \end{aligned}$$

$$|\psi_3\rangle = \frac{(|000\rangle - |001\rangle - |010\rangle + |100\rangle + |011\rangle - |111\rangle + |101\rangle - |110\rangle)}{2\sqrt{2}}$$