CSL202: Discrete Mathematical Structures
Tutorial/Homework: 09

1. James Bond is imprisoned in a cell from which there are three possible ways to escape: an air-conditioning duct, a sewer pipe and the door (which is unlocked). The air-conditioning duct leads him on a two-hour trip whereupon he falls through a trap door onto his head, much to the amusement of his captors. The sewer pipe is similar but takes five hours to traverse. Each fall produces temporary amnesia and he is returned to the cell immediately after each fall. Assume that he always immediately chooses one of the three exits from the cell with probability 1/3. On the average, how long does it take before he realizes that the door is unlocked and escapes?

2. k objects are picked independently at random with replacement from a set of n distinct objects. For $1 \leq i < j \leq k$, let X_{ij} denote the indicator random variable that is 1 if the i^{th} and j^{th} objects are the same otherwise 0. Show that for any $i < j$ and $p < q$ such that $(i, j) \neq (p, q)$, the random variables X_{ij} and X_{pq} are independent.

3. (**Coupon-collector problem**) Every time you go to the superstore, you get a random coupon out of n distinct coupons. What is the expected number of times you have to visit the store to be able to collect all distinct coupons?

4. (**Balls and bins**) n balls are thrown randomly into n bins. Let E be the event that no bin has more than $\frac{3 \ln n}{\ln \ln n}$ balls. Show that $\mathbf{Pr}[E] \geq (1 - 1/n)$.

5. (**Universal Hashing**) Hashing is a technique used to store elements from a large universe $U = \{0, ..., m-1\}$ using a small table $T = \{0, ..., n-1\}$ using a hash function $h : U \to T$ such that the number of collisions are minimized [1].

   Using a fixed hash function might does not work. So, we use a *family* of hash functions H and then pick a hash function randomly from this family. A hash function family H is called 2-universal if

   $$\forall x, y \in U, x \neq y, \mathbf{Pr}_{h \leftarrow H}[h(x) = h(y)] \leq 1/n.$$

   Show how a 2-universal hash function family is useful in hashing and give an example of such a family.

[1] Assume that collisions are resolved using auxiliary data structure