CSL202: Discrete Mathematical Structures
Tutorial/Homework: 07

1. A jigsaw puzzle is put together by successively joining pieces that fit together into blocks. A move is made each time a piece is added to a block, or when two blocks are joined. Use strong induction to prove that no matter how the moves are carried out, exactly $n-1$ moves are required to assemble a puzzle with n pieces.

2. Find the flaw with the following "proof" that $a^n = 1$ for all nonnegative integers n, whenever a is a nonzero real number.
   Basis step: $a^0 = 1$ is true by the definition of a^0.

   Inductive step: Assume that $a^j = 1$ for all nonnegative integers j with $j \leq k$. Then note that $a^{k+1} = \frac{a^k \cdot a^k}{a^{k-1}} = \frac{1 \cdot 1}{1} = 1$.

3. We will use the definition of a bipartite graph in this question:

   Definition 1.0.1 (Bipartite graph) *A bipartite graph is a graph where the vertices can be partitioned into two non-empty disjoint sets X, Y such that there are no edges between two vertices of X or two vertices of y (i.e., all edges are between a vertex in X and a vertex in Y). Bipartite graphs are typically defined as (X, Y, E), where X and Y denote the partitions and E denote the set of edges.*

   Given a Bipartite graph $G = (X, Y, E)$, a matching in G is defined to be a subset of $M \subseteq E$ such that for any $(x, y) \in M$, both x and y do not appear as the end vertex of any edge in M. Given $|X| = |Y| = n$, a perfect matching of G is a matching of size n. Show the following theorem:

   Theorem 1.0.2 (Hall's Theorem) *Given a bipartite graph $G = (X, Y, E)$ such that $n = |X| = |Y|$, there is a perfect matching in G is and only if for ever subset of vertices $S \subseteq X, |N(S)| \geq |S|$, where $N(S)$ denotes the set of neighboring vertices of S.*

4. Let S be the subset of the set of ordered pairs of integers defined recursively by:

   Basis step: $(0, 0) \in S$

   Recursive step: If $(a, b) \in S$, then $(a, b+1) \in S$, $(a+1, b+1) \in S$, and $(a+2, b+1) \in S$.

   (a) List the elements of S produced by the first four applications of the recursive definition.

   (b) Use strong induction on the number of applications of the recursive step of the definition to show that $a \leq 2b$ whenever $(a, b) \in S$.

   (c) Use structural induction to show that $a \leq 2b$ whenever $(a, b) \in S$.

5. Solve the following recurrence relation and write the exact value of $T(n)$.

$$T(n) = \begin{cases} T(n-1) & \text{if } n > 1 \text{ and } n \text{ is odd} \\ 2 \cdot T(n/2) & \text{if } n > 1 \text{ and } n \text{ is even} \\ 1 & \text{if } n = 1 \end{cases}$$

(*Hint: Unrolling may be tricky. You may want to guess the bound and then use induction to confirm your guess.*)