CSL202: Discrete Mathematical Structures

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Relations

Closure

A relation S on a set A is called the **closure** of another relation R on A with respect to property P if S has property P, S contains R, and S is a subset of every relation with property P containing R.

- Question: How do we find the transitive closure of any relation R on set A?

Theorem

Let R be a relation on a set A. There is a path of length n, where n is a positive integer, from a to b if and only if $(a, b) \in R^n$.

Definition (Connectivity relation)

Let R be a relation on a set A. The connectivity relation R^* consists of the pairs (a, b) such that there is a path of length at least one from a to b in R.

- Claim: $R^* = \cup_{n=1}^{\infty} R^n$.

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The transitive closure of a relation R equals the connectivity relation R^*.

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Lemma

Let A be a set with n elements and let R be a relation on A. If there is a path of length at least one in R from a to b, then there is such a path with length not exceeding n. Moreover, when $a \neq b$, if there is a path of length at least one in R from a to b, then there is such a path with length not exceeding $(n - 1)$.

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- <u>Claim</u>: $R^* = \cup_{i=1}^{n} R^i$.

- Question: How do we find the transitive closure of any relation R on set A?
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Theorem

Let M_R be the 0/1 matrix representing a relation R on a set of n elements. The the 0/1 matrix of the transitive closure R^* is

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee ... \vee M_R^{[n]}.$$

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- Example: Given $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, what is M_{R^*}?

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- Question: Can we find closure in fewer bit operations?
  - Warshall's Algorithm solves the problem in $O(n^3)$ bit operations.

Definition (Equivalence relation)

A relation in a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

Definition (Equivalent elements)

Two elements a and b that are related by an equivalence relation are called equivalent. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

- Question: Let $m > 1$ be an integer. Show that $R = \{(a, b) \mid a \equiv b \ (mod \ m)\}$ is an equivalence relation on the set of integers.

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Definition (Equivalence class)

Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the equivalence class of a. The equivalence class of a with respect to R is denoted by $[a]_R$. When only one relation is under consideration, we can delete the subscript R and write $[a]$ for this equivalence class.

- Question: What are the equivalence classes of 0 and 1 for congruence modulo 4?

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Theorem

Let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent: (i) $(a, b) \in R$, (ii) $[a] = [b]$, and (iii) $[a] \cap [b] \neq \emptyset$.

Theorem

Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i | i \in I\}$ of the set S, there is an equivalence relation R that has the sets $A_i, i \in I$, as its equivalence classes.

End