CSL202: Discrete Mathematical Structures

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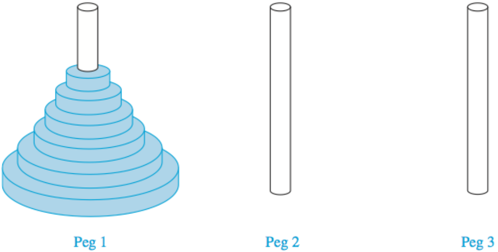
Advanced Counting Techniques

- Tower of Hanoi: Let H_n denote the number of moves needed to solve the Tower of Hanoi problem with n disks. Set up a recurrence relation for the sequence $\{H_n\}$.



- Find a recurrence relation and give initial conditions for the number of bit strings of length n that do not have two consecutive 0s. How many such bit strings are there of length five?

- Dynamic Programming: This is an algorithmic technique where a problem is recursively broken down into simpler overlapping subproblems, and the solution is computed using the solutions of the subproblems.
- Problem: Given a sequence of integers, find the length of the *longest increasing subsequence* of the given sequence.
    - Example: The longest increasing subsequence of the sequence $(7, 2, 8, 10, 3, 6, 9, 7)$ is $(2, 3, 6, 7)$ and its length is 4.

Definition (Linear homogeneous recurrence)

A *linear homogeneous* recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k},$$

where $c_1, c_2, ..., c_k$ are real numbers, and $c_k \neq 0$.

- *Linear* means that that RHS is a sum of linear terms of the previous elements of the sequence.
  - $a_n = a_{n-1} + a_{n-2}$ is a linear recurrence relation whereas $a_n = a_{n-1} + a_{n-2}^2$ is not.

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- *Linear* means that that RHS is a sum of linear terms of the previous elements of the sequence.
- *Homogeneous* means that there are no terms in the RHS that are not multiples of a_j's.
  - $a_n = a_{n-1} + a_{n-2}$ is homogeneous whereas $a_n = a_{n-1} + a_{n-2} + 2$ is not.

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- *Linear* means that that RHS is a sum of linear terms of the previous elements of the sequence.
- *Homogeneous* means that there are no terms in the RHS that are not multiples of a_j's.
- The coefficients of all the terms on the RHS are constants.
- The degree is k since a_n is expressed as the previous k terms of the sequence.

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where $c_1, c_2, ..., c_k$ are real numbers, and $c_k \neq 0$.

- $a_n = r^n$ is a solution of the recurrence if and only if

$$r^k - c_1 r^{k-1} - ... - c_k = 0. \tag{1}$$

- (1) is called the *characteristic equation* of the recurrence relation.
- The solutions of the characteristic equation are called the *characteristic roots* of the recurrence relation.

Theorem

Let c_1 and c_2 be real numbers. Suppose $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the linear homogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for all $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

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- What is the solution of the recurrence relation $a_n = a_{n-1} + 2 \cdot a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?

Theorem

Let c_1 and c_2 be real numbers. Suppose $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the linear homogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for all $n = 0, 1, 2, ...$, where α_1 and α_2 are constants.

Theorem

Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, for $n = 0, 1, 2, ...$, where α_1 and α_2 are constants.

- What is the solution of the recurrence relation $a_n = 6a_{n-1} - 9 \cdot a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?

Theorem

Let $c_1, c_2, ..., c_k$ be real numbers. Consider the linear homogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$. Suppose the characteristic equation of the recurrence relation has k distinct characteristic roots $r_1, r_2, ..., r_k$. Then $\{a_n\}$ is a solution of the recurrence relation if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + ... + \alpha_k r_k^n$ for $n = 0, 1, 2, ...,$ where $\alpha_1, \alpha_2, ..., \alpha_k$ are constants.

- What is the solution of the recurrence relation $a_n = 6a_{n-1} - 11 \cdot a_{n-2} + 6a_{n-3}$ with $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$?

Theorem

Let $c_1, c_2, ..., c_k$ be real numbers. Consider the linear homogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$. Suppose the characteristic equation of the recurrence relation has $t \leq k$ distinct characteristic roots $r_1, r_2, ..., r_t$ with multiplicities $m_1, m_2, ..., m_t$, respectively, so that $m_i \geq 1$ for $i = 1, 2, ..., t$ and $m_1 + m_2 + ... + m_t = k$. Then $\{a_n\}$ is a solution of the recurrence relation if and only if

$$\begin{aligned} a_n &= (\alpha_{1,0} + \alpha_{1,1} n + ... + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ &\quad + (\alpha_{2,0} + \alpha_{2,1} n + ... + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ &\quad + ... + (\alpha_{t,0} + \alpha_{t,1} n + ... + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

for $n = 0, 1, 2, ...$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

- What is the solution of the recurrence relation $a_n = -3a_{n-1} - 3 \cdot a_{n-2} - a_{n-3}$ with $a_0 = 1$, $a_1 = -2$, and $a_2 = -1$?

- A *linear non-homogeneous recurrence relation with constant coefficients* is a recurrence of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k} + F(n),$$

where $F(n)$ is a function not identically equal to zero and depending only on n.
- The recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$ is called the *associated homogeneous recurrence relation*.

Theorem

If $\{a_n^{(p)}\}$ is a particular solution of the non-homogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k} + F(n),$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}.$$

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- Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

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- Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?
- Findall solutions if the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$.

Theorem

Suppose $\{a_n\}$ satisfies the linear non-homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k} + F(n),$$

where $c_1, c_2, ..., c_k$ are real numbers, and

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + ... + b_1 n + b_0) s^n,$$

where $b_0, b_1, ..., b_t$ are s real numbers. When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + ... + p_1 n + p_0) s^n.$$

When s is a root of this characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + ... + p_1 n + p_0) s^n.$$

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Theorem

Let f be an increasing function that satisfies the recurrence relation

$$f(n) = a \cdot f(n/b) + c$$

whenever n is divisible by b, where $a \geq 1$, b is an integer greater than 1, and c is a positive real number. Then

$$f(n) \ is \begin{cases} O(n^{\log_b a}) & if\ a > 1 \\ O(\log n) & if\ a = 1 \end{cases}$$

Furthermore, when $n = b^k$ and $a \neq 1$, where k is a positive integer, $f(n) = C_1 n^{\log_b a} + C_2$, where $C_1 = f(1) + c/(a-1)$ and $C_2 = -c/(a-1)$.

Theorem (Master Theorem)

Let f be an increasing function that satisfies the recurrence relation

$$f(n) = a \cdot f(n/b) + cn^d$$

whenever $n = b^k$, where k is a positive integer, $a \geq 1$, b is an integer greater than 1, and c and d are real numbers with c positive and d nonnegative. Then

$$f(n) \textit{ is } \begin{cases} O(n^d) & \textit{if } a < b^d \\ O(n^d \log n) & \textit{if } a = b^d \\ O(n^{\log_b a}) & \textit{if } a > b^d. \end{cases}$$

Advanced Counting Techniques: Generating Functions

Theorem (Generating function)

The generating function for the sequence $a_0, a_1, ..., a_k, ...$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + ... + a_k x^k + ... = \sum_{k=0}^{\infty} a_k x^k$$

- We can define generating functions for finite sequences of real numbers by extending a finite sequence $a_0, a_1, ..., a_n$ into an infinite sequence by setting $a_{n+1} = 0$, $a_{n+2} = 0$, and so on.
- Examples:
  - What is the generating function for the sequence $1, 1, 1, 1, 1, 1$?
  - Let m be a positive integer and let $a_k = \binom{m}{k}$, for $k = 0, 1, ..., m$. What is the generating function for $a_0, a_1, ..., a_m$?

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  - What is the generating function for the sequence $1, 1, 1, 1, 1, 1$?
  - Let m be a positive integer and let $a_k = \binom{m}{k}$, for $k = 0, 1, ..., m$. What is the generating function for $a_0, a_1, ..., a_m$?
  - The function $f(x) = \frac{1}{1-x}$ is the generating function of the sequence $1, 1, ...$, because $\frac{1}{1-x} = 1 + x + x^2 + ...$ for $|x| < 1$.

Theorem

Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ *and* $g(x) = \sum_{k=0}^{\infty} b_k x^k$*. Then* $f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$ *and* $f(x)g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) x^k$*.*

- Let $f(x) = \frac{1}{(1-x)^2}$. Find coefficients $a_0, a_1, ...$ in the expansion $f(x) = \sum_{k=0}^{\infty} a_k x^k$.

Definition (Extended binomial coefficient)

Let u be a real number and k a nonnegative integer. Then the extended binomial coefficient $\binom{u}{k}$ is defined by

$$\binom{u}{k} = \begin{cases} \frac{u(u-1)...(u-k+1)}{k!} & \text{if } k > 0 \\ 1 & \text{if } k = 0 \end{cases}$$

- Find the value of the extended binomial coefficient $\binom{1/2}{3}$.
- Find the value of the extended binomial coefficient $\binom{-n}{r}$.

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Theorem (Extended binomial theorem)

Let x be a real number with $|x| < 1$ and let u be a real number. Then

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k.$$

- What is the expansion of $(1-x)^{-n}$?

TABLE 1 Useful Generating Functions.

$G(x)$	a_k
$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k = 1 + C(n,1)x + C(n,2)x^2 + \cdots + x^n$	$C(n,k)$
$(1+ax)^n = \sum_{k=0}^{n} C(n,k)a^k x^k = 1 + C(n,1)ax + C(n,2)a^2x^2 + \cdots + a^n x^n$	$C(n,k)a^k$
$(1+x^r)^n = \sum_{k=0}^{n} C(n,k)x^{rk} = 1 + C(n,1)x^r + C(n,2)x^{2r} + \cdots + x^{rn}$	$C(n,k/r)$ if $r \mid k$; 0 otherwise
$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$	1 if $k \le n$; 0 otherwise
$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$	1
$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$	a^k
$\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots$	1 if $r \mid k$; 0 otherwise
$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + \cdots$	$k+1$
$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k = 1 + C(n,1)x + C(n+1,2)x^2 + \cdots$	$C(n+k-1,k) = C(n+k-1,n-1)$
$\frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)(-1)^k x^k = 1 - C(n,1)x + C(n+1,2)x^2 - \cdots$	$(-1)^k C(n+k-1,k) = (-1)^k C(n+k-1,n-1)$
$\frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)a^k x^k = 1 + C(n,1)ax + C(n+1,2)a^2x^2 + \cdots$	$C(n+k-1,k)a^k = C(n+k-1,n-1)a^k$
$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$1/k!$
$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$(-1)^{k+1}/k$

- In how many different ways can eight identical cookies be distributed among three distinct children if each child receives at least two cookies and no more than four cookies?
- Use generating functions to determine the number of ways to insert tokens worth \$1, \$2, and \$5 into a vending machine to pay for an item that costs r dollars in both the cases when the order in which the tokens are inserted does not matter and when the order does matter.
- Use generating functions to find the number of r-combinations from a set with n elements when repetition of elements is allowed.

Solve the recurrence relation $a_k = 3a_{k-1}$ for $k = 1, 2, ...$ and initial condition $a_0 = 2$.

- Let $G(x)$ be the generating function for the sequence $\{a_k\}$.
- Claim 1: $xG(x) = \sum_{k=1}^{\infty} a_{k-1}x^k$.
- Claim 2: $G(x) - 3xG(x) = a_0$.
- Claim 3: $G(x) = \sum_{k=0}^{\infty} 2 \cdot 3^k \cdot x^k$.

Advanced counting techniques: Inclusion-Exclusion

Theorem (The Principle of Inclusion-Exclusion)

Let $A_1, A_2, ..., A_n$ be finite sets. Then

$$|A_1 \cup A_2 \cup ... \cup A_n| = \sum_{1 \leq i \neq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - ... + (-1)^{n+1} |A_1 \cap A_2 \cap ... \cap A_n|.$$

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- **The Hatcheck Problem** A new employee checks the hats of n people at a restaurant, forgetting to put claim check numbers on the hats. When customers return for their hats, the checker gives them back hats chosen at random from the remaining hats. What is the probability that no one receives the correct hat?

End