# CSL202: Discrete Mathematical Structures

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# Discrete Probability

- Probabilistic algorithms: Algorithms that make random choices at one or more steps.
- Monte Carlo Algorithms: Probabilistic algorithms for decision problems that always produces an answer. The answer may be incorrect with some small probability.
  - Example: $A$ sends 1 million apples to $B$. $A$ has cleverly packed 1000 bad apples among these 1 million apples. How does $B$ detect that $A$ has sent 1 million good apples or not.

**Theorem (The Probabilistic Method)**

*If the probability that an element chosen at random from a $S$ does not have a particular property is less than 1, there exists an element in $S$ with this property.*

- An existence proof based on the probabilistic method is nonconstructive because it does not find a particular element with the desired property.

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- Example: *Ramsey number*
  - Assume that in a group of six people, each pair of individuals consists of two friends or two enemies. Show that there are either three mutual friends or three mutual enemies in the group.

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- Example: *Ramsey number*
  - The Ramsey number $R(m, n)$, where $m$ and $n$ are positive integers greater than or equal to 2, denotes the minimum number of people at a party such that there are either m mutual friends or $n$ mutual enemies, assuming that every pair of people at the party are friends or enemies.

**Theorem (The Probabilistic Method)**

*If the probability that an element chosen at random from a S does not have a particular property is less than 1, there exists an element in S with this property.*

**Definition (Ramsey number)**

The Ramsey number $R(m, n)$, where $m$ and $n$ are positive integers greater than or equal to 2, denotes the minimum number of people at a party such that there are either m mutual friends or $n$ mutual enemies, assuming that every pair of people at the party are friends or enemies.

**Theorem**

*If k is an integer with $k \geq 2$, then $R(k, k) \geq 2^{k/2}$.*

**Theorem (Baye's Theorem)**

*Suppose that $E$ and $F$ are events from a sample space $S$ such that $\mathbf{Pr}[E] \neq 0$ and $\mathbf{Pr}[F] \neq 0$. Then*

$$\mathbf{Pr}[F|E] = \frac{\mathbf{Pr}[E|F] \cdot \mathbf{Pr}[F]}{\mathbf{Pr}[E|F] \cdot \mathbf{Pr}[F] + \mathbf{Pr}[E|\bar{F}] \cdot \mathbf{Pr}[\bar{F}]}.$$

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- Example: We have two boxes. The first contains two green balls and seven red balls; the second contains four green balls and three red balls. Bob selects a ball by first choosing one of the two boxes at random. He then selects one of the balls in this box at random. If Bob has selected a red ball, what is the probability that he selected a ball from the first box?

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- Example: Suppose that one person in 100,000 has a particular rare disease for which there is a fairly accurate diagnostic test. This test is correct 99.0% of the time when given to a person selected at random who has the disease; it is correct 99.5% of the time when given to a person selected at random who does not have the disease. Given this information can we find
  (a) the probability that a person who tests positive for the disease has the disease?
  (b) the probability that a person who tests negative for the disease does not have the disease?

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- Other Application: Bayesian spam filtering.

**Theorem (Generalized Baye's Theorem)**

*Suppose that $E$ is an event from a sample space $S$ and that $F_1, ..., F_n$ are mutually exclusive events such that $\cup_{i=1}^{n} F_i = S$. Assume that $\mathbf{Pr}[E] \neq 0$ and $\mathbf{Pr}[F_i] \neq 0$ for $i = 1, 2, ..., n$. Then*

$$\mathbf{Pr}[F_j|E] = \frac{\mathbf{Pr}[E|F_j] \cdot \mathbf{Pr}[F_j]}{\sum_{i=1}^{n} \mathbf{Pr}[E|F_i] \cdot \mathbf{Pr}[F_i]}.$$

End