CSL202: Discrete Mathematical Structures

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Counting

Theorem (The Pigeonhole Principle)

If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Corollary

A function f from a set with $k+1$ or more elements to a set with k elements is not one-to-one.

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- Show that for every integer n there is a multiple of n that has only 0s and 1s in its decimal expansion.

Theorem (The Generalized Pigeonhole Principle)

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

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- What is the least number of area codes needed to guarantee that the 25 million phones in a state can be assigned distinct 10-digit telephone numbers? (Assume that telephone numbers are of the form NXX-NXX-XXXX, where the first three digits form the area code, N represents a digit from 2 to 9 inclusive, and X represents any digit.)

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- Suppose that a computer science laboratory has 15 workstations and 10 servers. A cable can be used to directly connect a workstation to a server. For each server, only one direct connection to that server can be active at any time. We want to guarantee that at any time any set of 10 or fewer workstations can simultaneously access different servers via direct connections. What is the minimum number of direct connections needed to achieve this goal?

- During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.
- Show that among any $n+1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.
- Every sequence of n^2+1 distinct real numbers contains a subsequence of length $n+1$ that is either strictly increasing or strictly decreasing.

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Permutation and Combination

Definition (Permutation)

A *permutation* of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an r-permutation.

Theorem

If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n,r) = n(n-1)(n-2)...(n-r+1) = \frac{n!}{(n-r)!}$$

r-permutations of a set with n distinct elements.

- How many permutations of the letters ABCDEFGH contain the string ABC ?

Definition (Combination)

An r-combination of elements of a set is an unordered selection of r elements from the set. Thus, an r-combination is simply a subset of the set with r elements.
The number of r-combinations of a set with n distinct elements is denoted by $C(n, r)$. Note that $C(n, r)$ is also denoted by $\binom{n}{r}$ and is called a binomial coefficient.

Theorem

The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \leq r \leq n$, equals

$$C(n, r) = \frac{n!}{r!(n-r)!}$$

- How many ways are there to select 47 cards from a standard deck of 52 cards?

Theorem

Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.

- There is a simple algebraic proof of this theorem. Another way is to give a *combinatorial proof*.

Definition

A *combinatorial proof* of an identity is a proof that uses counting arguments to prove that both sides of the identity count the same objects but in different ways or a proof that is based on showing that there is a bijection between the sets of objects counted by the two sides of the identity. These two types of proofs are called *double counting proofs* and *bijective proofs*, respectively.

- Example: How many bit strings of length n contain exactly r 1s?

- The number of r-combinations from a set with n elements is denoted by $\binom{n}{r}$.

Theorem (The Binomial theorem)

Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j.$$

- Exercises:
  - Let n be a nonnegative integer. Show that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.
  - Let n be a nonnegative integer. Show that $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.
  - Let n be a nonnegative integer. Show that $\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$.

Theorem (Pascal's Identity)

Let n and k be positive integers with $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Theorem (Vandermonde's Identity)

Let m, n, and r be nonnegative integers with r not exceeding m and n. Then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}.$$

- Exercise: Show that if n is a nonnegative integer, then $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$.

Theorem

Let n and r be nonnegative integers with $r \leq n$. Then

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}.$$

Theorem (Permutation with repetition)

The number of r-permutations of a set of n objects with repetition allowed is n^r.

- Example: How many strings of length r can be formed from the uppercase letters of the English alphabet?

Theorem (Combination with repetition)

There are $C(n+r-1, r) = C(n+r-1, n-1)$ r-combinations from a set with n elements when repetition of elements is allowed.

- Example: Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen? Assume that only the type of cookie, and not the individual cookies or the order in which they are chosen, matters.
- How many solutions does the equation $x_1 + x_2 + x_3 = 11$ have, where x_1, x_2, and x_3 are nonnegative integers?

Theorem (Permutation with indistinguishable objects)

The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, ..., and n_k indistinguishable objects of type k, is

$$\frac{n!}{n_1!n_2!...n_k!}.$$

- Example: How many different strings can be made by reordering the letters of the word SUCCESS?

Theorem (Distinguishable objects into distinguishable boxes)

The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, $i = 1, 2, ..., k$, equals

$$\frac{n!}{n_1!n_2!...n_k!}$$

- In how many ways can you place n indistinguishable objects into k distinguishable boxes?

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  - This is the same as generating permutations of $\{1, 2, ..., n\}$.
- Total ordering on permutations of $\{1, 2, 3, ..., n\}$:
  - $(a_1, a_2, ..., a_n) < (b_1, b_2, ..., b_n)$ iff there is a j such that $a_1 = b_1, a_2, = b_2, ..., a_{j-1} = b_{j-1}$, and $a_j < b_j$.
- Question: What is the next permutation after $(a_1, a_2, ..., a_n)$?

- How do you generate a combination of n distinct objects?

End