# CSL202: Discrete Mathematical Structures

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# Cardinality of Sets

**Definition**

The sets $A$ and $B$ have the same cardinality if there is a one-to-one correspondence from $A$ to $B$. When $A$ and $B$ have the same cardinality, we write $|A| = |B|$.

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If there is a one-to-one function from $A$ to $B$, the cardinality of $A$ is less than or the same as the cardinality of $B$ and we write $|A| \leq |B|$. The cardinality of $A$ is less than the cardinality of $B$, written as $|A| < |B|$, if there is an injection but no surjection from $A$ to $B$.

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  1. Claim 1: There is an injection from $S$ to $\mathcal{P}(S)$.
     - Consider a function $f : S \rightarrow \mathcal{P}(S)$ defined as: for any $s \in S, f(s) = \{s\}$. This is an injective function.
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  2. Claim 2: There is no surjection from $S$ to $\mathcal{P}(S)$.
     - Consider any function $f : S \rightarrow \mathcal{P}(S)$ and consider the following set defined in terms of this function: $A = \{x | x \notin f(x)\}$
     - Claim 2.1: There does not exist an element $s \in S$ such that $f(s) = A$.

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     - Claim 2.1: There does not exist an element $s \in S$ such that $f(s) = A$.
     - Proof: For the sake of contradiction, assume that there is an $s \in S$ such that $f(s) = A$. The following bi-implications follow:

$$\begin{aligned} s \in A &\leftrightarrow s \in \{x | x \notin f(x)\} \\ &\leftrightarrow s \notin f(s) \\ &\leftrightarrow s \notin A \end{aligned}$$

This is a contradiction. Hence the statement of the claim holds. □

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**Definition (Countable and uncountable sets)**

A set that is either finite or has the same cardinality as the set of positive integers is called *countable*. A set that is not countable is called *uncountable*.

End