CSL202: Discrete Mathematical Structures

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Definition (Logical equivalence)

Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain is used for the variables in these propositional functions. We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

- Are these logically equivalent:
  - $\neg \forall x P(x)$ and $\exists x \neg P(x)$?
  - $\neg \exists x P(x)$ and $\forall x \neg P(x)$?

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- These are logically equivalent:
  - $\neg \forall x P(x)$ and $\exists x \neg P(x)$
  - $\neg \exists x P(x)$ and $\forall x \neg P(x)$
- These rules for negation of quantifiers are called *De Morgan's laws for quantifiers*.
- Show that $\neg \forall x (P(x) \rightarrow Q(x))$ and $\exists x (P(x) \wedge \neg Q(x))$ are logically equivalent.

- Analyze complex natural language sentences.
  - Example: "*Every student in this class has visited either Delhi or Mumbai.*"
- Translate system specifications.
  - Example: "*Every mail message larger than one megabyte will be compressed.*"
- Deriving conclusions from statements:
  - Examples: Consider the following statements
    - "*All lions are fierce.*"
    - "*Some lions do not drink coffee.*"
  - From the above two sentences can we make the following conclusion?
    - "*Some fierce creatures do not drink coffee.*"

- Nested Quantifiers: Two quantifiers are nested if one is within the scope of the other.
  - Example:
    - $\forall x \exists y (x + y = 0)$.
    - We may write the above as $\forall x Q(x)$, where $Q(x) = \exists y (x + y = 0)$.
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- The order of the quantifiers is important. Consider the following examples:
  - Let $Q(x, y)$ denote $(x + y = 0)$ and let the domain for x, y consist of all real numbers.
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Statement	When True	When False
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	?	?
$\forall x \exists y P(x,y)$	?	?
$\exists x \forall y P(x,y)$	?	?
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Table: Nested quantification of two variables.

Statement	When True	When False
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	$P(x,y)$ is True for every pair x,y	There is a pair x,y for which $P(x,y)$ is False
$\forall x \exists y P(x,y)$	For every x there is a y such that $P(x,y)$ is True	There is an x such that $P(x,y)$ is False for every y
$\exists x \forall y P(x,y)$	There is an x for which $P(x,y)$ is True for every y	For every x there is a y for which $P(x,y)$ is False.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair x,y for which $P(x,y)$ is True	$P(x,y)$ is False for every pair x,y

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- Determine the truth value of these statements when the domain of all variables consists of all real numbers and $Q(x, y, z)$ denotes $x + y = z$.
  - $\forall x \forall y \exists z Q(x, y, z)$?
  - $\exists z \forall x \forall y Q(x, y, z)$?

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    - $\forall x \forall y \exists z Q(x, y, z)$? True
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- Translating mathematical statements into logical expressions involving nested quantifiers.
    - "*The sum of two positive integers is always positive.*"
    - "*Every real number except 0 has a multiplicative inverse.*"

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- Translating mathematical statements into logical expressions involving nested quantifiers.
  - "*The sum of two positive integers is always positive.*"
    - $\forall x \forall y((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$
  - "*Every real number except 0 has a multiplicative inverse.*"
    - $\forall x((x \neq 0) \rightarrow \exists y(xy = 1))$

- Translating nested quantifiers into English.
  - Example:
    - $F(a, b)$: "*a and b are friends.*"
    - The domain consists of all students in the Institute.
    - Translate: $\exists x \forall y \forall z (F(x, y) \wedge F(x, z) \wedge (y \neq z) \rightarrow \neg(F(y, z)))$

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    - "*There is a student none of whose friends are also friends with each other.*"

- Translating English sentences into logical expressions:
- Consider following examples:
  - Translate: "*If a person is female and is a parent, then this person is someone's mother.*"
  - Translate: "*Everyone has exactly one best friend.*"
  - Translate: "*There does not exist a woman who has taken a flight on every airline in the world.*"

End