CSL202: Discrete Mathematical Structures

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Definition (Tautology and Contradiction)

A compound proposition that is always true, no matter what the truth values of the proposition that occurs in it, is called a tautology. A compound proposition that is always false is called a contradiction. A compound proposition that is neither a tautology nor a contradiction is called a contingency.

- Examples:
  - $(p \vee \neg p)$ is a tautology.
  - $(p \wedge \neg p)$ is a contradiction.

Definition (Logical equivalence)

A compound proposition p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

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- Show that $\neg(p \wedge q) \equiv \neg p \vee \neg q$.

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Equivalence	Name
$p \wedge T \equiv ?$ $p \vee F \equiv ?$	Identity laws
$p \vee T \equiv ?$ $p \wedge F \equiv ?$	Domination laws
$p \vee p \equiv ?$ $p \wedge p \equiv ?$	Idempotent laws
$\neg(\neg p) \equiv ?$	Double negation law
$p \vee q \equiv ?$ $p \wedge q \equiv ?$	Commutative laws
$(p \vee q) \vee r \equiv ?$ $(p \wedge q) \wedge r \equiv ?$	Associative laws
$p \vee (q \wedge r) \equiv ?$ $p \wedge (q \vee r) \equiv ?$	Distributive laws

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- Show that $\neg(p \rightarrow q) \equiv (p \wedge \neg q)$.
- Show that $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$.
- Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

- Consider the following two statements:
  - "*All computers connected to the Institute network are functioning properly.*"
  - "*Computer-1 is connected to the Institute network.*"
- Is it ok to make the conclusion that Computer-1 is functioning properly?

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- Suppose there are only two computers in the institute. Consider the following propositions:
  - p: Computer-1 is connected to the network.
  - q: Computer-2 is connected to the network.
  - r: Computer-1 is functioning properly.
  - s: Computer-2 is functioning properly.
- We can write $(p \rightarrow r) \wedge (q \rightarrow s) \wedge p$.

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- Suppose there are only two computers on the institute. Consider the following propositions:
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- Now, suppose there are 10,000 computers in the institute?

Predicate Logic

- Consider the following two statements:
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    - "*Computer-1 is connected to the Institute network.*"
- Is it ok to make the conclusion that Computer-1 is functioning properly?
- Suppose there are 10,000 computers in the institute?
- Consider the following concise way of writing propositions:
    - $P(x)$: x is connected to the institute network.
        - x can take values Computer-1, Computer-2 etc.
        - P denotes the *predicate* "is connected to the institute network."
        - $P(x)$ can be thought of the value of the *propositional function* P at x.

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- Consider the following concise way of writing propositions:
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- Consider the following concise way of writing propositions:
  - $P(x)$: x is connected to the institute network.
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- Are $P(x)$ and $R(x)$ propositions? No, but $P(\text{Computer-100})$ and $R(\text{Computer-200})$ are propositions.

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- What we would like to say is that for any assignment of x from the set {Computer-1, ..., Computer-10000}, $P(x) \rightarrow R(x)$.

- Quantification expresses the extent to which a predicate is true over a range of elements.
- There are two types of quantification:
  - *Universal quantification* which tells that a predicate is true for every element under consideration.
  - *Existential quantification* tells us that there is one or more element under consideration for which the predicate is true.
- The area of logic that deals with predicates and quantifiers is called *predicate calculus*.

Definition (Universal quantification)

The universal quantification of $P(x)$ is the statement "$P(x)$ for all values of x in the *domain*." The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the *universal quantifier*. We read $\forall x P(x)$ as "for all x $P(x)$." An element for which $P(x)$ is false is called a *counterexample* of $\forall x P(x)$.

- Examples:
  - Let $P(x) : x + 1 > x$. The truth value of the quantification $\forall x P(x)$ is true when the domain consists of all real numbers.

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- Examples:
  - Let $P(x) : x + 1 > x$. The truth value of the quantification $\forall x P(x)$ is true when the domain consists of all real numbers.
  - Let $P(x) : x^2 > 0$. What is the truth value of $\forall x P(x)$ when the domain consists of all integers?

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Definition (Existential quantification)

The existential quantification of $P(x)$ is the statement "there exists an element x in the domain such that $P(x)$." We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the existential quantifier.

- Examples:
    - Let $P(x) : x^2 \leq 0$. What is the truth value of $\exists x P(x)$ when the domain consists of all integers?

- Quantifiers with restricted domain:
  - What does the following mean when the domain consists of all real numbers:
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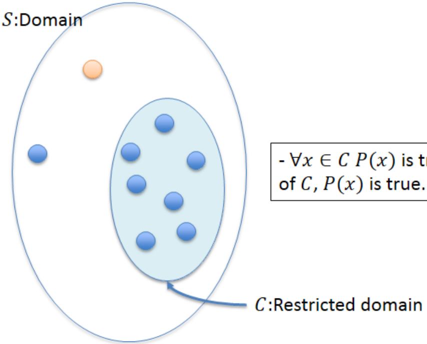

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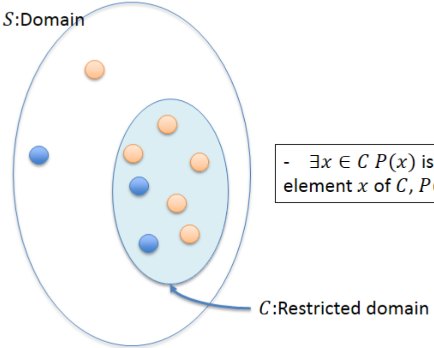

- $\exists x \in C \; P(x)$ is true when for at least one element x of C, $P(x)$ is true.

- Quantifiers with restricted domain:
  - What does the following mean when the domain consists of all real numbers:
    - $\forall x < 0(x^2 > 0)$: $\forall x(x > 0 \rightarrow x^2 > 0)$
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- More definitions: Binding and free variables, scope.
    - Binding variable: When a quantifier is used on a variable x, we say that this occurence of the variable is *bound*.
    - Free variable: An occurence of a variable that is not bound by a quantifier or set equal to a particular value is said to be *free*.
    - Scope of variable: The part of a logical expression to which a quantifier is applied is called the *scope* of this quantifier.

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  - Examples:
    - $\exists x (x + y = 1)$
    - $\forall x (P(x) \wedge Q(x)) \vee \forall x R(x)$

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Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain is used for the variables in these propositional functions. We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

- Are these logically equivalent:
    - $\forall x(P(x) \wedge Q(x))$ and $\forall x P(x) \wedge \forall x Q(x)$?
    - $\exists x(P(x) \vee Q(x))$ and $\exists x P(x) \vee \exists x Q(x)$?
    - $\forall x(P(x) \vee Q(x))$ and $\forall x P(x) \vee \forall x Q(x)$?
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- These are logically equivalent:
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  - $\neg\exists x P(x)$ and $\forall x \neg P(x)$
- These rules for negation of quantifiers are called *De Morgan's laws for quantifiers*.
- Show that $\neg\forall x(P(x) \rightarrow Q(x))$ and $\exists x(P(x) \wedge \neg Q(x))$ are logically equivalent.

- Analyze complex natural language sentences.
  - Example: "*Every student in this class has visited either Delhi or Mumbai.*"
- Translate system specifications.
  - Example: "*Every mail message larger than one megabyte will be compressed.*"
- Deriving conclusions from statements:
  - Examples: Consider the following statements
    - "*All lions are fierce.*"
    - "*Some lions do not drink coffee.*"
  - From the above two sentences can we make the following conclusion?
    - "*Some fierce creatures do not drink coffee.*"

- Nested Quantifiers: Two quantifiers are nested if one is within the scope of the other.
  - Example:
    - $\forall x \exists y (x + y = 0)$.
    - We may write the above as $\forall x Q(x)$, where $Q(x) = \exists y (x + y = 0)$.
    - What does the above statement say when the domain for both variables consists of all real numbers?

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- The order of the quantifiers is important. Consider the following examples:
  - Let $Q(x, y)$ denote $(x + y = 0)$ and let the domain for x, y consist of all real numbers.
  - What is the truth value of $\exists y \forall x Q(x, y)$?

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Statement	When True	When False
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	?	?
$\forall x \exists y P(x, y)$	?	?
$\exists x \forall y P(x, y)$	?	?
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	?	?

Table: Nested quantification of two variables.

Statement	When True	When False
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	$P(x,y)$ is True for every pair x,y	There is a pair x,y for which $P(x,y)$ is False
$\forall x \exists y P(x,y)$	For every x there is a y such that $P(x,y)$ is True	There is an x such that $P(x,y)$ is False for every y
$\exists x \forall y P(x,y)$	There is an x for which $P(x,y)$ is True for every y	For every x there is a y for which $P(x,y)$ is False.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair x,y for which $P(x,y)$ is True	$P(x,y)$ is False for every pair x,y

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