

Name: \_\_\_\_\_

ID number: \_\_\_\_\_

There are 1 questions for a total of 10 points.

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1. (10 points) Consider the following randomized algorithm. The inputs are a positive integer  $n$  and an integer array  $A$  containing  $n$  distinct integers.

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FindMax( $A, n$ )
- Randomly permute the array  $A$ 
-  $Max \leftarrow A[1]$ 
- For  $i = 2$  to  $n$ 
  - if ( $A[i] > Max$ )  $Max \leftarrow A[i]$ 
- return( $Max$ )

```

What is the expected number of times the value of the variable  $Max$  changes within the for loop? Express your answer as a function of  $n$  using  $\Theta$  notation. Show details of your calculations.

**Solution:** Let  $X_i$  be an indicator random variable that is 1 if the value of  $Max$  gets modified in iteration  $i$  and 0 otherwise. The value of  $Max$  changes in the  $i^{th}$  iteration iff  $A[i]$  is the maximum element in the subarray  $A[1], \dots, A[i]$ . This happens with probability  $\frac{1}{i}$ . So, we have for all  $i = 2, 3, \dots, n$

$$\Pr[X_i = 1] = \frac{1}{i} \quad \text{and} \quad \mathbf{E}[X_i] = 1 \cdot \Pr[X_i = 1] = \frac{1}{i}$$

The number of times the variable  $Max$  changes is given by  $\sum_{i=2}^n X_i$ . The expectation of this quantity is given by:

$$\begin{aligned} \mathbf{E}\left[\sum_{i=2}^n X_i\right] &= \sum_{i=2}^n \mathbf{E}[X_i] \quad (\text{by linearity of expectation}) \\ &= \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \\ &= \Theta(\log n) \quad (\text{using discussion in class}) \end{aligned}$$