Name: ______________________________

ID number: ______________________________

There are 3 questions for a total of 10 points.

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1. Five swimmers training together either swam in a race or watched the others swim.

   (a) (1 point) At least how many races must have been scheduled if every swimmer had opportunity to watch all of the others?

   (a) **4**

   (b) (4 points) Explain your answer in part (a).

   Solution: The total number of *sightings* is $4 \times 5 = 20$. The maximum number of sightings in a race is $2 \times 3 = 6$ (when 3 people race and 2 observe or vice-versa). So, at least 4 races are required for all possible sightings. Next, we will show that 4 races are sufficient by describing the four races:
   Race 1: A, B, C
   Race 2: A, D, E
   Race 3: B, D
   Race 4: C, E

2. (3 points) What is the conditional probability that exactly three heads appear when a fair coin is flipped five times, given that the first flip came up tails? Write the final answer here and show your calculations in the space below.

2. ____1/4____

Solution: Let E be the event that exactly three heads appear and let F be the event that the first flip comes up tails. We have $\mathbf{Pr}[F] = 1/2$ and $\mathbf{Pr}[E \cap F] = 4/32 = 1/8$. The latter is because $E \cap F = \{THHHT, THHTH, THTHH, TTHHH\}$. So, $\mathbf{Pr}[E|F] = \frac{\mathbf{Pr}[E \cap F]}{\mathbf{Pr}[F]} = \frac{(1/8)}{(1/2)} = 1/4$.

3. (2 points) What is the minimum number of balls you need to throw randomly into 15 bins so that with probability at least 1/2, there is a bin with at least two balls. Write the final answer here and show your calculations in the space below.

3. ____5____

Solution: Let E_t denote the event that all the t balls end up in distinct bins when t balls are thrown. Then we have:

$$\mathbf{Pr}[E_t] = 1 \cdot \frac{14}{15} \cdot \frac{13}{15} \dots \frac{15 - t + 1}{15}$$

The probability that there is a bin with at least two balls is equal to $1 - \mathbf{Pr}[E_t]$. The minimum value of t that makes $\mathbf{Pr}[E_t] \leq 1/2$ (and hence $1 - \mathbf{Pr}[E_t] \geq 1/2$) is $t = 5$. This is because $\mathbf{Pr}[E_5] < 0.5$ and $\mathbf{Pr}[E_4] > 0.5$.