COL202: Discrete Mathematical Structures
Tutorial/Homework: 13

1. Discuss Quiz-10 questions.
2. Complete discussion of Tutorial-12 problems in case needed.
3. How many ways are there for a horse race with four horses to finish if ties are possible?
4. Prove the *Hockeystick identity*

$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$$

whenever n and r are positive integers,

   (a) using a combinatorial argument.
   (b) using Pascal's identity.

5. Give a combinatorial proof that $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$.
6. Give a combinatorial proof that $\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$.
7. Suppose that a weapons inspector must inspect each of five different sites twice, visiting one site per day. The inspector is free to select the order in which to visit these sites, but cannot visit site X, the most suspicious site, on two consecutive days. In how many different orders can the inspector visit these sites?
8. Consider the following two theorems:

   Theorem 13.0.1 (Permutation with indistinguishable objects) *The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, ..., and n_k indistinguishable objects of type k, is*

$$\frac{n!}{n_1!n_2!...n_k!}.$$

   Theorem 13.0.2 (Distinguishable objects into distinguishable boxes) *The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, $i = 1, 2, ..., k$, equals*

$$\frac{n!}{n_1!n_2!...n_k!}$$

Prove the second theorem by first setting up a one-to-one correspondence between permutations of n objects with n_i indistinguishable objects of type $i, i = 1, 2, ..., k$ and the distribution of n objects in k boxes such that n_i objects are placed in box $i, i = 1, 2, ..., k$ and then applying the first theorem.

9. *Monty Hall three-door puzzle*: You are in a game show where there are three doors. There is prize behind only one of the doors which you do not know. The game show host asks you to pick a door. Once you pick a door, the host opens a door behind which there is no prize and then asks whether you would like to change the choice of your door or continue with your first choice. Use probabilities to decide whether you should change the door or continue with the first one.

10. Suppose that instead of three doors, there are four doors in the Monty Hall puzzle. What is the probability that you win by not changing once the host, who knows what is behind each door, opens a losing door and gives you the chance to change doors? What is the probability that you win by changing the door you select to one of the two remaining doors among the three that you did not select?

11. Which is more likely: rolling a total of 8 when two dice are rolled or rolling a total of 8 when three dice are rolled?

12. What is the probability of these events when we randomly select a permutation of $\{1, 2, ..., n\}$ where $n \geq 4$?

    a) 1 precedes 2.

    b) 2 precedes 1.

    c) 1 immediately precedes 2.

    d) n precedes 1 and n-1 precedes 2.

    c) n precedes 1 and n precedes 2.

13. What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up tails?

14. James Bond is imprisoned in a cell from which there are three possible ways to escape: an air-conditioning duct, a sewer pipe and the door (which is unlocked). The air-conditioning duct leads him on a two-hour trip whereupon he falls through a trap door onto his head, much to the amusement of his captors. The sewer pipe is similar but takes five hours to traverse. Each fall produces temporary amnesia and he is returned to the cell immediately after each fall. Assume that he always immediately chooses one of the three exits from the cell with probability 1/3. On the average, how long does it take before he realizes that the door is unlocked and escapes?