COL202: Discrete Mathematical Structures

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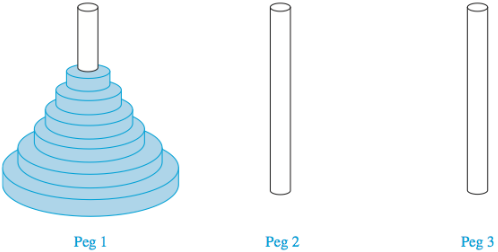
Advanced Counting Techniques

- Tower of Hanoi: Let H_n denote the number of moves needed to solve the Tower of Hanoi problem with n disks. Set up a recurrence relation for the sequence $\{H_n\}$.



- Find a recurrence relation and give initial conditions for the number of bit strings of length n that do not have two consecutive 0s. How many such bit strings are there of length five?

- Dynamic Programming: This is an algorithmic technique where a problem is recursively broken down into simpler overlapping subproblems, and the solution is computed using the solutions of the subproblems.
- Problem: Given a sequence of integers, find the length of the *longest increasing subsequence* of the given sequence.
  - Example: The longest increasing subsequence of the sequence $(7, 2, 8, 10, 3, 6, 9, 7)$ is $(2, 3, 6, 7)$ and its length is 4.

Definition (Linear homogeneous recurrence)

A *linear homogeneous* recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k},$$

where $c_1, c_2, ..., c_k$ are real numbers, and $c_k \neq 0$.

- *Linear* means that that RHS is a sum of linear terms of the previous elements of the sequence.
  - $a_n = a_{n-1} + a_{n-2}$ is a linear recurrence relation whereas $a_n = a_{n-1} + a_{n-2}^2$ is not.

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- *Linear* means that that RHS is a sum of linear terms of the previous elements of the sequence.
- *Homogeneous* means that there are no terms in the RHS that are not multiples of a_j's.
  - $a_n = a_{n-1} + a_{n-2}$ is homogeneous whereas $a_n = a_{n-1} + a_{n-2} + 2$ is not.

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A *linear homogeneous* recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

- *Linear* means that that RHS is a sum of linear terms of the previous elements of the sequence.
- *Homogeneous* means that there are no terms in the RHS that are not multiples of a_j's.
- The coefficients of all the terms on the RHS are constants.
- The degree is k since a_n is expressed as the previous k terms of the sequence.

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where $c_1, c_2, ..., c_k$ are real numbers, and $c_k \neq 0$.

- $a_n = r^n$ is a solution of the recurrence if and only if

$$r^k - c_1 r^{k-1} - ... - c_k = 0. \quad (1)$$

- (1) is called the *characteristic equation* of the recurrence relation.
- The solutions of the characteristic equation are called the *characteristic roots* of the recurrence relation.

Theorem

Let c_1 and c_2 be real numbers. Suppose $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the linear homogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for all $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

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- What is the solution of the recurrence relation $a_n = a_{n-1} + 2 \cdot a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?

Theorem

Let c_1 and c_2 be real numbers. Suppose $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the linear homogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for all $n = 0, 1, 2, ...$, where α_1 and α_2 are constants.

Theorem

Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, for $n = 0, 1, 2, ...$, where α_1 and α_2 are constants.

- What is the solution of the recurrence relation $a_n = 6a_{n-1} - 9 \cdot a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?

Theorem

Let $c_1, c_2, ..., c_k$ be real numbers. Consider the linear homogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$. Suppose the characteristic equation of the recurrence relation has k distinct characteristic roots $r_1, r_2, ..., r_k$. Then $\{a_n\}$ is a solution of the recurrence relation if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + ... + \alpha_k r_k^n$ for $n = 0, 1, 2, ...,$ where $\alpha_1, \alpha_2, ..., \alpha_k$ are constants.

- What is the solution of the recurrence relation $a_n = 6a_{n-1} - 11 \cdot a_{n-2} + 6a_{n-3}$ with $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$?

Theorem

Let $c_1, c_2, ..., c_k$ be real numbers. Consider the linear homogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$. Suppose the characteristic equation of the recurrence relation has $t \leq k$ distinct characteristic roots $r_1, r_2, ..., r_t$ with multiplicities $m_1, m_2, ..., m_t$, respectively, so that $m_i \geq 1$ for $i = 1, 2, ..., t$ and $m_1 + m_2 + ... + m_t = k$. Then $\{a_n\}$ is a solution of the recurrence relation if and only if

$$\begin{aligned} a_n &= (\alpha_{1,0} + \alpha_{1,1} n + ... + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ &+ (\alpha_{2,0} + \alpha_{2,1} n + ... + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ &+ ... + (\alpha_{t,0} + \alpha_{t,1} n + ... + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

for $n = 0, 1, 2, ...,$ where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

- What is the solution of the recurrence relation $a_n = -3a_{n-1} - 3 \cdot a_{n-2} - a_{n-3}$ with $a_0 = 1$, $a_1 = -2$, and $a_2 = -1$?

- A *linear non-homogeneous recurrence relation with constant coefficients* is a recurrence of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k} + F(n),$$

where $F(n)$ is a function not identically equal to zero and depending only on n.
- The recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$ is called the *associated homogeneous recurrence relation*.

Theorem

If $\{a_n^{(p)}\}$ is a particular solution of the non-homogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k} + F(n),$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}.$$

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- Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

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- Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?
- Findall solutions if the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$.

Theorem

Suppose $\{a_n\}$ satisfies the linear non-homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k} + F(n),$$

where $c_1, c_2, ..., c_k$ are real numbers, and

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + ... + b_1 n + b_0) s^n,$$

where $b_0, b_1, ..., b_t$ are s real numbers. When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + ... + p_1 n + p_0) s^n.$$

When s is a root of this characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + ... + p_1 n + p_0) s^n.$$

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Theorem

Let f be an increasing function that satisfies the recurrence relation

$$f(n) = a \cdot f(n/b) + c$$

whenever n is divisible by b, where $a \geq 1$, b is an integer greater than 1, and c is a positive real number. Then

$$f(n) \ is \begin{cases} O(n^{\log_b a}) & \text{if } a > 1 \\ O(\log n) & \text{if } a = 1 \end{cases}$$

Furthermore, when $n = b^k$ and $a \neq 1$, where k is a positive integer, $f(n) = C_1 n^{\log_b a} + C_2$, where $C_1 = f(1) + c/(a-1)$ and $C_2 = -c/(a-1)$.

Theorem (Master Theorem)

Let f be an increasing function that satisfies the recurrence relation

$$f(n) = a \cdot f(n/b) + cn^d$$

whenever $n = b^k$, where k is a positive integer, $a \geq 1$, b is an integer greater than 1, and c and d are real numbers with c positive and d nonnegative. Then

$$f(n) \text{ is } \begin{cases} O(n^d) & \text{if } a < b^d \\ O(n^d \log n) & \text{if } a = b^d \\ O(n^{\log_b a}) & \text{if } a > b^d. \end{cases}$$

Advanced Counting Techniques: Generating Functions

Theorem (Generating function)

The generating function for the sequence $a_0, a_1, ..., a_k, ...$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + ... + a_k x^k + ... = \sum_{k=0}^{\infty} a_k x^k$$

- We can define generating functions for finite sequences of real numbers by extending a finite sequence $a_0, a_1, ..., a_n$ into an infinite sequence by setting $a_{n+1} = 0$, $a_{n+2} = 0$, and so on.
- Examples:
  - What is the generating function for the sequence $1, 1, 1, 1, 1, 1$?
  - Let m be a positive integer and let $a_k = \binom{m}{k}$, for $k = 0, 1, ..., m$. What is the generating function for $a_0, a_1, ..., a_m$?

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- Examples:
  - What is the generating function for the sequence $1, 1, 1, 1, 1, 1$?
  - Let m be a positive integer and let $a_k = \binom{m}{k}$, for $k = 0, 1, ..., m$. What is the generating function for $a_0, a_1, ..., a_m$?
  - The function $f(x) = \frac{1}{1-x}$ is the generating function of the sequence $1, 1, ...,$ because $\frac{1}{1-x} = 1 + x + x^2 + ...$ for $|x| < 1$.

Theorem

Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then $f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$ and $f(x)g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) x^k$.

- Let $f(x) = \frac{1}{(1-x)^2}$. Find coefficients $a_0, a_1, ...$ in the expansion $f(x) = \sum_{k=0}^{\infty} a_k x^k$.

Definition (Extended binomial coefficient)

Let u be a real number and k a nonnegative integer. Then the extended binomial coefficient $\binom{u}{k}$ is defined by

$$\binom{u}{k} = \begin{cases} \frac{u(u-1)\dots(u-k+1)}{k!} & \text{if } k > 0 \\ 1 & \text{if } k = 0 \end{cases}$$

- Find the value of the extended binomial coefficient $\binom{1/2}{3}$.
- Find the value of the extended binomial coefficient $\binom{-n}{r}$.

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Theorem (Extended binomial theorem)

Let x be a real number with $|x| < 1$ and let u be a real number. Then

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k.$$

- What is the expansion of $(1-x)^{-n}$?

TABLE 1 Useful Generating Functions.

$G(x)$	a_k
$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k = 1 + C(n,1)x + C(n,2)x^2 + \cdots + x^n$	$C(n,k)$
$(1+ax)^n = \sum_{k=0}^{n} C(n,k)a^k x^k = 1 + C(n,1)ax + C(n,2)a^2x^2 + \cdots + a^n x^n$	$C(n,k)a^k$
$(1+x^r)^n = \sum_{k=0}^{n} C(n,k)x^{rk} = 1 + C(n,1)x^r + C(n,2)x^{2r} + \cdots + x^{rn}$	$C(n,k/r)$ if $r \mid k$; 0 otherwise
$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$	1 if $k \le n$; 0 otherwise
$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$	1
$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$	a^k
$\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots$	1 if $r \mid k$; 0 otherwise
$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + \cdots$	$k+1$
$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k = 1 + C(n,1)x + C(n+1,2)x^2 + \cdots$	$C(n+k-1,k) = C(n+k-1,n-1)$
$\frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)(-1)^k x^k = 1 - C(n,1)x + C(n+1,2)x^2 - \cdots$	$(-1)^k C(n+k-1,k) = (-1)^k C(n+k-1,n-1)$
$\frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)a^k x^k = 1 + C(n,1)ax + C(n+1,2)a^2x^2 + \cdots$	$C(n+k-1,k)a^k = C(n+k-1,n-1)a^k$
$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$1/k!$
$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$(-1)^{k+1}/k$

- In how many different ways can eight identical cookies be distributed among three distinct children if each child receives at least two cookies and no more than four cookies?
- Use generating functions to determine the number of ways to insert tokens worth \$1, \$2, and \$5 into a vending machine to pay for an item that costs r dollars in both the cases when the order in which the tokens are inserted does not matter and when the order does matter.
- Use generating functions to find the number of r-combinations from a set with n elements when repetition of elements is allowed.

End