COL202: Discrete Mathematical Structures

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Induction and Recursion

- The process of defining objects in terms of itself is called *recursion*.
- Examples
  - Recursively defined functions.
  - Recursively defined sets and structures

- Recursively defined functions.
  - Functions with non-negative integers as its domain.
  - Basis step: Specify the value of the function at 0.
  - Recursive step: Give a rule for finding its value at an integer from its values at smaller integers.
  - Such functions may alternatively be interpreted as a sequence.
- Example: Fibonacci sequence
  - Basis step: $f_0 = 0, f_1 = 1$.
  - Recursive step: $f_n = f_{n-1} + f_{n-2}$ for $n = 2, 3, 4, ...$
  - Show that whenever $n \geq 3, f_n > \alpha^{n-2}$, where $\alpha = (1 + \sqrt{5})/2$.

- Recursively defined sets and structures
  - Basis step: An initial collection of elements is specified.
  - Recursive step: Rules for forming new elements in the set from those already known to be in the set are provided.
  - *Exclusion Rule*: Recursively defined set contains nothing other than those elements specified in the basis step or generated by applications of the recursive step.
- Examples:
  - Multiples of 3:
    - Basis step: $3 \in S$
    - Recursive step: If $x \in S$ and $y \in S$, then $x + y \in S$.
  - The set Σ^* of strings over the alphabet Σ:
    - Basis step: ?
    - Recursive step: ?

- Examples:
  - Multiples of 3:
    - Basis step: $3 \in S$
    - Recursive step: If $x \in S$ and $y \in S$, then $x + y \in S$.
  - The set Σ^* of strings over the alphabet Σ:
    - Basis step: $\lambda \in \Sigma^*$ (where λ is the empty string containing no symbols).
    - Recursive step: If $w \in \Sigma^*$ and $x \in \Sigma$, then $wx \in \Sigma^*$.
  - Well-formed formula in propositional logic:
    - Basis step: T, F, and s, where s is a propositional variable, are well-formed formulae.
    - Recursive step: If E and F are well-formed formulae, then $(\neg E)$, $(E \wedge F)$, $(E \vee F)$, $(E \rightarrow F)$, and $(E \leftrightarrow F)$ are well-formed formulae.

- More examples:
  - Rooted trees:
    - Basis step: A single vertex r is a rooted tree.
    - Recursive step: Suppose that $T_1, T_2, ..., T_n$ are disjoint rooted trees with roots $r_1, r_2, ..., r_n$, respectively. Then the graph formed by starting with a root r, which is not in any of the rooted trees $T_1, T_2, ..., T_n$, and adding an edge from r to each of the vertices $r_1, r_2, ..., r_n$, is also a rooted tree.
  - Full binary trees:
    - Basis step: There is a full binary tree consisting only of a single vertex r.
    - Recursive step: If T_1 and T_2 are disjoint full binary trees, there is a full binary tree, denoted by $T_1 \cdot T_2$, consisting of a root r together with edges connecting the root to each of the roots of the left subtree T_1 and the right subtree T_2.

- A set S defined recursively:
  - Basis step: $3 \in S$
  - Recursive step: If $x \in S$ and $y \in S$, then $x + y \in S$.
- Let A be the set of all positive integers divisible by 3.
- Show that $S \subseteq A$.
- *Structural Induction*
  - Basis step: Show that the result holds for all elements specified in the basis step of the recursive definition to be in the set.
  - Recursive step: Show that if the statement is true for each of the elements used to construct new elements in the recursive step of the definition, the result holds for these new elements.
- Argue the validity of structural induction.

- Well-formed formula in propositional logic:
  - Basis step: T, F, and s, where s is a propositional variable, are well-formed formulae.
  - Recursive step: If E and F are well-formed formulae, then $(\neg E)$, $(E \wedge F)$, $(E \vee F)$, $(E \rightarrow F)$, and $(E \leftrightarrow F)$ are well-formed formulae.
- Show that every well-formed formula for compound propositions, as defined above, contains an equal number of left and right parentheses.

Definition

An algorithm is called *recursive* if it solves a problem by reducing it to an instance of the same problem with smaller input.

- Example:

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Extended-Euclid-GCD(a, b)
  If(b = 0), then return(a, 1, 0)
  else
    Compute integers q, r such that a = qb + r and 0 ≤ r < b.
    Let (d, x, y) = Extended-Euclid-GCD(b, r)
    return(d, y, x − yq)
```

- Prove that the above algorithm returns the correct answer.

Counting

- What are we going to count?
  - Objects with certain properties.
- Basic Counting Principles:
  - The Product Rule: Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 \cdot n_2$ ways to do the procedure.

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  - Extended Product Rule: Suppose that a procedure is carried out by performing the tasks $T_1, T_2, ..., T_m$ in sequence. If each task $T_i, i = 1, 2, ..., n$, can be done in n_i ways, regardless of how the previous tasks were done, then there are $n_1 \cdot n_2 ... n_m$ ways to carry out the procedure.
  - Can you argue the validity of the extended product rule using the product rule?

- Basic Counting Principles:
  - The Product Rule: Examples
    - How many functions are there from a set of m elements to a set of n elements?
    - How many such one-one functions are there?

- Basic Counting Principles:
  - The Product Rule:
  - The Sum Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.
  - The Extended Sum Rule: Suppose that a task can be done in one of n_1 ways, in one of n_2 ways,...,or in one of n_m ways, where none of the set of n_i ways of doing the task is the same as any of the set of n_j ways, for all pairs i and j with$1 \leq i < j \leq m$. Then the number of ways to do the task is $n_1 + n_2 + ... + n_m$.

- Basic Counting Principles:
  - The Product Rule:
  - The Sum Rule: Examples
    - Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

- Basic Counting Principles:
  - The Product Rule:
  - The Sum Rule:
  - The Subtraction Rule: If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.
    - The subtraction rule is also known as the principle of *inclusion-exclusion*, especially when it is used to count the number of elements in the union of two sets.
    - $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$.

- Basic Counting Principles:
  - The Product Rule:
  - The Sum Rule:
  - The Subtraction Rule:
  - The Division Rule: If f is a function from A to B where A and B are finite sets, and that for every value $y \in B$ there are exactly d values $x \in A$ such that $f(x) = y$ (in which case, we say that f is d-to-one), then $|B| = |A|/d$.
    - Example: How many different ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left neighbor and the same right neighbor?

- Basic Counting Principles:
  - The Product Rule
  - The Sum Rule
  - The Subtraction Rule
  - The Division Rule

- Tree diagrams: We use a branch to represent each possible choice. The possible outcomes are represented by the *leaves*.
  - Example: A playoff between two teams consists of at most five games. The first team that wins three games wins the playoff. In how many different ways can the playoff occur?

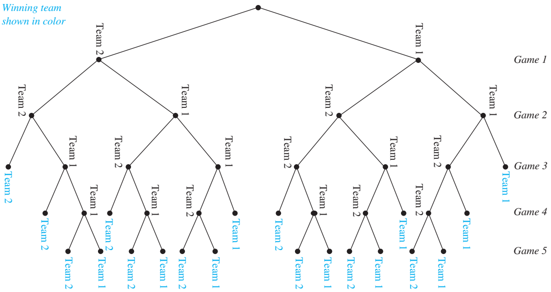

FIGURE 3 **Best Three Games Out of Five Playoffs.**

Theorem (The Pigeonhole Principle)

If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Corollary

A function f from a set with $k+1$ or more elements to a set with k elements is not one-to-one.

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- Show that for every integer n there is a multiple of n that has only 0s and 1s in its decimal expansion.

Theorem (The Generalized Pigeonhole Principle)

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

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- What is the least number of area codes needed to guarantee that the 25 million phones in a state can be assigned distinct 10-digit telephone numbers? (Assume that telephone numbers are of the form NXX-NXX-XXXX, where the first three digits form the area code, N represents a digit from 2 to 9 inclusive, and X represents any digit.)

Theorem (The Generalized Pigeonhole Principle)

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

- Suppose that a computer science laboratory has 15 workstations and 10 servers. A cable can be used to directly connect a workstation to a server. For each server, only one direct connection to that server can be active at any time. We want to guarantee that at any time any set of 10 or fewer workstations can simultaneously access different servers via direct connections. What is the minimum number of direct connections needed to achieve this goal?

- During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.
- Show that among any $n + 1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.
- Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.

End