COL202: Discrete Mathematical Structures

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Cardinality of Sets

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The sets A and B have the same cardinality if there is a one-to-one correspondence from A to B. When A and B have the same cardinality, we write |A| = |B|.

Definition

If there is a one-to-one function from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \le |B|$. The cardinality of A is less than the cardinality of B, written as |A| < |B|, if there is an injection but no surjection from A to B.

Definition (Countable and uncountable sets)

A set that is either finite or has the same cardinality as the set of positive integers is called *countable*. A set that is not countable is called *uncountable*.

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• Show that the set of odd positive integers is a countable set.

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Let S be a set. Then $|S| < \mathcal{P}(S)$.

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Let S be a set. Then $|S| < \mathcal{P}(S)$.

Proof sketch

- We need to show the following:
 - **1** <u>Claim 1</u>: There is an injection from S to $\mathcal{P}(S)$.
 - **2** <u>Claim 2</u>: There is no surjection from S to $\mathcal{P}(S)$.

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- We need to show the following:
 - **1** <u>Claim 1</u>: There is an injection from S to $\mathcal{P}(S)$.
 - Consider a function $f: S \rightarrow \mathcal{P}(S)$ defined as: for any
 - $s \in S, f(s) = \{s\}$. This is an injective function.
 - **2** <u>Claim 2</u>: There is no surjection from S to $\mathcal{P}(S)$.

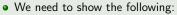
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- **2** <u>Claim 2</u>: There is no surjection from S to $\mathcal{P}(S)$.
 - Consider any function f : S → P(S) and consider the following set defined in terms of this function: A = {x|x ∉ f(x)}
 - <u>Claim 2.1</u>: There does not exist an element $s \in S$ such that f(s) = A.

Basic Structures Cardinality of Sets

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 - Claim 2: There is no surjection from S to P(S).
 - Consider any function f : S → P(S) and consider the following set defined in terms of this function: A = {x|x ∉ f(x)}
 - Claim 2.1: There does not exist an element $s \in S$ such that f(s) = A.
 - Proof: For the sake of contradiction, assume that there is an s ∈ S such that f(s) = A. The following bi-implications follow:

$$s \in A \quad \leftrightarrow \quad s \in \{x | x \notin f(x) \\ \leftrightarrow \quad s \notin f(s) \\ \leftrightarrow \quad s \notin A$$

This is a contradiction. Hence the statement of the claim holds.

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- Show that the set of all integers is countable.

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- Show that the set of positive rational numbers is countable.

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- An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers).
- Show that the set of all integers is countable.
- Show that the set of positive rational numbers is countable.
- Show that the set of real number is an uncountable set.

- Which of the following statements true:
 - Every integer has a finite size description in decimal.
 - Every real number has a finite size description in decimal.
 - Every rational number has a finite size description in decimal

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- Where does the diagonalization argument fail in case of integers?

• If A and B are countable sets, then $A \cup B$ is also countable.

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Theorem (Schröder-Bernstein theorem)

If there are one-to-one functions f from A to B and g from B to A, then there is a one-to-one correspondence between A and B.

• Show that |(0,1)| = |(0,1]|.

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