COL202: Discrete Mathematical Structures

Ragesh Jaiswal, CSE, IIT Delhi

Cardinality of Sets

Definition

The sets A and B have the same cardinality if there is a one-to-one correspondence from A to B. When A and B have the same cardinality, we write $|A| = |B|$.

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     - Consider a function $f : S \to \mathcal{P}(S)$ defined as: for any $s \in S, f(s) = \{s\}$. This is an injective function.
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  2. Claim 2: There is no surjection from S to $\mathcal{P}(S)$.
     - Consider any function $f : S \rightarrow \mathcal{P}(S)$ and consider the following set defined in terms of this function: $A = \{x | x \notin f(x)\}$
     - Claim 2.1: There does not exist an element $s \in S$ such that $f(s) = A$.

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     - Claim 2.1: There does not exist an element $s \in S$ such that $f(s) = A$.
     - Proof: For the sake of contradiction, assume that there is an $s \in S$ such that $f(s) = A$. The following bi-implications follow:

$$\begin{aligned} s \in A \quad &\leftrightarrow \quad s \in \{x | x \notin f(x)\} \\ &\leftrightarrow \quad s \notin f(s) \\ &\leftrightarrow \quad s \notin A \end{aligned}$$

This is a contradiction. Hence the statement of the claim holds. □

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- Show that the set of all integers is countable.
- Show that the set of positive rational numbers is countable.
- Show that the set of real number is an uncountable set.

- Which of the following statements true:
  - Every integer has a finite size description in decimal.
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- Where does the diagonalization argument fail in case of integers?

- If A and B are countable sets, then $A \cup B$ is also countable.

Theorem (Schröder-Bernstein theorem)

If there are one-to-one functions f from A to B and g from B to A, then there is a one-to-one correspondence between A and B.

- Show that $|(0,1)| = |(0,1]|$.

End