COL202: Discrete Mathematical Structures

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Definition (Argument and argument form)

An *argument* in propositional logic is a sequence of propositions. All but the final proposition in the argument are called *premises* and the final proposition is called the *conclusion*. An argument is valid if the truth of all its premises implies that the conclusion is true.

An *argument form* in propositional logic is a sequence of compound propositions involving propositional variables. An argument form is *valid* if no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.

- How do we show that an argument form is valid?

Definition (Argument and argument form)

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An *argument form* in propositional logic is a sequence of compound propositions involving propositional variables. An argument form is *valid* if no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.

- How do we show that an argument form is valid?
  - Construct a truth table. However, this could be tedious.
- We first show the validity of some simple argument forms. These are called *rules of inference*. These may be used to show the validity of more complex argument forms.

Rule of inference	Tautology	Name
p $p \to q$ $\therefore ?$	$[p \wedge (p \to q)] \to ?$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg ?$	$[\neg q \wedge (p \to q)] \to ?$	Modus tollens
$p \to q$ $q \to r$ $\therefore ?$	$[(p \to q) \wedge (q \to r)] \to ?$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore ?$	$[(p \vee q) \wedge \neg p] \to ?$	Disjunctive syllogism

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$p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$[(p \vee q) \wedge \neg p] \rightarrow (q)$	Disjunctive syllogism

Table: Rules of inference.

Rule of inference	Tautology	Name
$\dfrac{p}{\therefore ?}$	$p \rightarrow ?$	Addition
$\dfrac{p \wedge q}{\therefore ?}$	$(p \wedge q) \rightarrow ?$	Simplification
p $\dfrac{q}{\therefore ?}$	$[(p) \wedge (q)] \rightarrow ?$	Conjunction
$p \vee q$ $\dfrac{\neg p \vee r}{\therefore ?}$	$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow ?$	Resolution

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$\dfrac{p \quad q}{\therefore p \wedge q}$	$[(p) \wedge (q)] \rightarrow (p \wedge q)$	Conjunction
$\dfrac{p \vee q \quad \neg p \vee r}{\therefore q \vee r}$	$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$	Resolution

Table: Rules of inference.

- Which rule of inference is used in the argument:

  If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have barbecue tomorrow.

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  If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have barbecue tomorrow.

- Show that the hypothesis:
  - "*It is not sunny this afternoon and it is colder than yesterday*."
  - "*We will go swimming only if it is sunny*."
  - "*If we do not go swimming, then we will take a canoe trip*."
  - "*If we take a canoe trip, then we will be home by sunset*."
- lead to the conclusion:
  - "*We will be home by sunset*."

- Show that the following argument is valid. If today is tuesday, I have a test in Mathematics or economics. If my Economics Professor is sick, I will not have a test in Economics. Today is tuesday and my Economics Professor is sick. Therefore I have a test in Mathematics.
- Show that the following argument is valid. If Mohan is a lawyer, then he is ambitious. If Mohan is an early riser, then he does not like idlies. If Mohan is ambitious, then he is an early riser. Then if Mohan is a lawyer, then he does not like idlies.

Resolution Principle

- *Resolution Principle* is another way of showing that an argument is correct.
- Definitions:
  - Literal: A variable or a negation of a variable is called a literal.
  - Sum and Product: A disjunction of literals is called a sum and a conjunction of literals is called a product.
  - Clause: A disjunction of literals is called a clause.
  - Resolvent: For any two clauses C_1 and C_2, if there is a literal L_1 in C_1 that is complementary to literal L_2 in C_2, then delete L_1 and L_2 from C_1 and C_2 respectively and construct the disjunction of the remaining clauses. The constructed clause is a resolvent of C_1 and C_2.
    - $C_1 = P \vee Q \vee R$
    - $C_2 = \neg P \vee \neg S \vee T$
    - What is a resolvent of C_1 and C_2?

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    - $C_1 = P \vee Q \vee R$
    - $C_2 = \neg P \vee \neg S \vee T$
    - What is a resolvent of C_1 and C_2? $Q \vee R \vee \neg S \vee T$

Theorem

Given two clauses C_1 and C_2, a resolvent C of C_1 and C_2 is a logical consequence of C_1 and C_2.

- Example: Modus ponens $(P \wedge (P \rightarrow Q) \rightarrow Q)$
  - C_1: P
  - C_2: $\neg P \vee Q$
  - The resolvent of C_1 and C_2 is Q which is a logical consequence of C_1 and C_2.

Theorem

Given two clauses C_1 and C_2, a resolvent C of C_1 and C_2 is a logical consequence of C_1 and C_2.

Definition (Resolution principle and refutation)

Given a set S of clauses, a (resolution) deduction of C from S is a finite sequence $C_1, ..., C_k$ of clauses such that each C_i either is a clause in S or a resolvent of clauses preceding C and $C_k = C$. A deduction of $\Box$ (empty clause) is called a *refutation*.

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Given a set S of clauses, a (resolution) deduction of C from S is a finite sequence $C_1, ..., C_k$ of clauses such that each C_i either is a clause in S or a resolvent of clauses preceding C and $C_k = C$. A deduction of $\square$ (empty clause) is called a *refutation* of a proof of S.

- If there is an argument where $P_1, ..., P_r$ are the premises and C is the conclusion, to get a proof using resolution principle, put $P_1, ..., P_r$ in clause form and add to it $\neg C$ in clause form. From this sequence, if $\square$ can be derived, the argument is valid.

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- Example:

$$\begin{array}{l} T \rightarrow (M \vee E) \\ S \rightarrow \neg E \\ T \wedge S \\ \hline \therefore M \end{array}$$

  - What are the clauses?

- Example:

$$\begin{array}{l} T \rightarrow (M \vee E) \\ S \rightarrow \neg E \\ T \wedge S \\ \hline \therefore M \end{array}$$

  - C_1: $\neg T \vee M \vee E$
  - C_2 : $\neg S \vee \neg E$
  - C_3 : T
  - C_4 : S
  - C_5 : $\neg M$

- Example:

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  - C_1: $\neg T \vee M \vee E$
  - C_2 : $\neg S \vee \neg E$
  - C_3 : T
  - C_4 : S
  - C_5 : $\neg M$
  - C_6 : $\neg T \vee M \vee \neg S$ (resolvent of C_1 and C_2)

- Example:

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  - C_1: $\neg T \vee M \vee E$
  - C_2 : $\neg S \vee \neg E$
  - C_3 : T
  - C_4 : S
  - C_5 : $\neg M$
  - C_6 : $\neg T \vee M \vee \neg S$ (resolvent of C_1 and C_2)
  - C_7 : $M \vee \neg S$ (resolvent of C_3 and C_6)

- Example:

$$\begin{array}{l} T \rightarrow (M \vee E) \\ S \rightarrow \neg E \\ T \wedge S \\ \hline \therefore M \end{array}$$

- C_1: $\neg T \vee M \vee E$
- C_2 : $\neg S \vee \neg E$
- C_3 : T
- C_4 : S
- C_5 : $\neg M$
- C_6 : $\neg T \vee M \vee \neg S$ (resolvent of C_1 and C_2)
- C_7 : $M \vee \neg S$ (resolvent of C_3 and C_6)
- C_8 : M (resolvent of C_4 and C_7)

- Example:

$$\begin{array}{l} T \rightarrow (M \vee E) \\ S \rightarrow \neg E \\ T \wedge S \\ \hline \therefore M \end{array}$$

  - C_1: $\neg T \vee M \vee E$
  - C_2 : $\neg S \vee \neg E$
  - C_3 : T
  - C_4 : S
  - C_5 : $\neg M$
  - C_6 : $\neg T \vee M \vee \neg S$ (resolvent of C_1 and C_2)
  - C_7 : $M \vee \neg S$ (resolvent of C_3 and C_6)
  - C_8 : M (resolvent of C_4 and C_7)
  - C_9 : $\square$ (resolvent of C_5 and C_8)
- Hence, from the resolution principle, the argument is valid.

Rules of Inference for Quantified Statements

Rule of inference	Name
$\forall x\ P(x)$ $\therefore ?$	Universal instantiation
$P(c)$ for an arbitrary c $\therefore ?$	Universal generalization
$\exists x\ P(x)$ $\therefore ?$	Existential instantiation
$P(c)$ for some element c $\therefore ?$	Existential generalization
$\forall x(P(x) \rightarrow Q(x))$ $P(a)$ where a is a particular element in the domain $\therefore ?$	Universal modus ponens
$\forall x(P(x) \rightarrow Q(x))$ $\neg Q(a)$ where a is a particular element in the domain $\therefore ?$	Universal modus tollens

Table: Rules of inference for quantified statements

Rule of inference	Name
$\forall x\ P(x)$ $\therefore P(c)$	Universal instantiation
$P(c)$ for an arbitrary c $\therefore \forall x\ P(x)$	Universal generalization
$\exists x\ P(x)$ $\therefore P(x)$ for some element c	Existential instantiation
$P(c)$ for some element c $\therefore \exists x\ P(x)$	Existential generalization
$\forall x(P(x) \rightarrow Q(x))$ $P(a)$ where a is a particular element in the domain $\therefore Q(a)$	Universal modus ponens
$\forall x(P(x) \rightarrow Q(x))$ $\neg Q(a)$ where a is a particular element in the domain $\therefore \neg P(a)$	Universal modus tollens

Table: Rules of inference for quantified statements

- Use rules of inference for quantified statements to show the premises “*A student in this class has not read the book,*” and “*Everyone in this class passed the first exam*” imply the conclusion “*Someone who has passed the first exam has not read the book.*”

End