# Multiplicative Rank-1 Approximation using Length-Squared Sampling

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[Joint work with Amit Kumar (IIT Delhi)]

# Main Result

## Main result (informal)

The span of $\Omega(\frac{1}{\varepsilon^4})$ rows of any matrix $A \subset \mathbb{R}^{n \times d}$ sampled according to the length-squared distribution, contains a rank-1 matrix that gives a $(1 + \varepsilon)$ **multiplicative** approximation under the Frobenius norm.

- Length-squared distribution: The probability of sampling the $i^{th}$ row, $\mathbf{A}^{(i)}$, is proportional to $\|\mathbf{A}^{(i)}\|^2$.

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$$\mathbf{E}[\|A - \tilde{A}\|_F^2] \leq (1+\varepsilon) \cdot \|A - \pi_1(A)\|_F^2, \quad \textit{where}$$

$$\pi_1(A) = \underset{X:rank(X)=1}{\operatorname{argmin}} \|A - X\|_F^2$$

# Discussion: Best Fit Line

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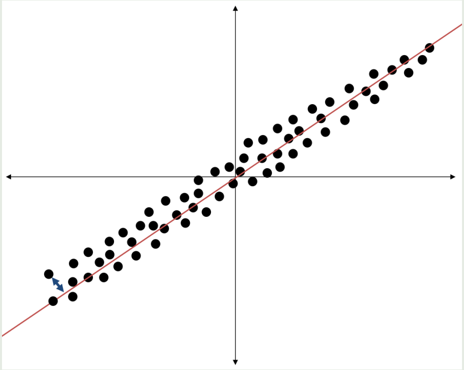
## Best Fit Line Problem (interpret rows as points)



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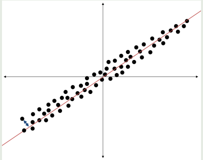

- The problem is **not** hard.
- The solution is the first singular vector and can be found using SVD.
- This discussion is about simple sampling based technique.

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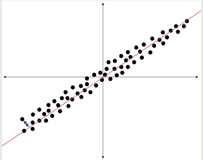

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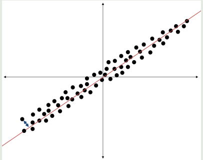

- Question: Can we approximate the best-fit line using a few samples?
  - Non-constructive: Yes using $O(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon})$ samples [SV12].

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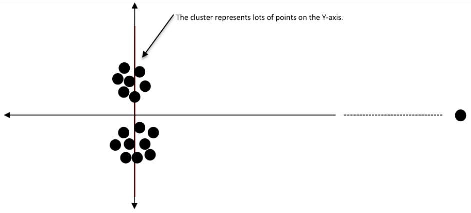
- Question: Can we approximate the best-fit line using a few **uniformly** sampled points?
  - No. See example below.



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## Best Fit Line Problem (interpret rows as points)

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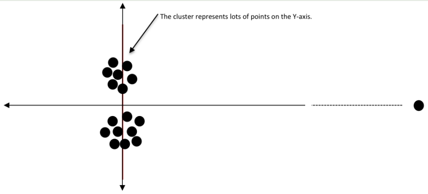
  - No. See example below.
  - This motivates length-squared sampling since the distance of a point from origin is relevant.



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**Theorem (Freize, Kannan, and Vempala [FKV04])**

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$$\mathbf{E}[\|A - \tilde{A}\|_F^2] \leq \|A - \pi_1(A)\|_F^2 \; + \; \varepsilon \cdot \|A\|_F^2.$$

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  - Yes. *Adaptive length-squared sampling* along with *volume sampling* gives multiplicative approximation with $O(\frac{1}{\varepsilon})$ samples [DV06, DRVW06].
  - The above even works for rank-$k$ approximation.

# Discussion: Our Result

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The span of $\Omega(\frac{1}{\varepsilon^4})$ rows of any matrix $A \subset \mathbb{R}^{n\times d}$ sampled according to the length-squared distribution, contains a rank-1 matrix that gives a $(1+\varepsilon)$ **multiplicative** approximation under the Frobenius norm.

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    - The above even works for rank-$k$ approximation.
- **This work**: Does length-squared sampling suffice for multiplicative rank-1 approximation? Yes with $O(\frac{1}{\varepsilon^4})$ samples.

# Discussion: Main Ideas

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- Let $\sigma^2 \equiv \|\pi_1(A)\|_F^2$ and $r^2 \equiv \|A - \pi_1(A)\|_F^2$.
- By a suitable rotation, we can assume that $\pi_1(A) = \begin{pmatrix} \sigma u_1, 0, \dots, 0 \\ \vdots \\ \sigma u_2, 0, \dots, 0 \end{pmatrix}$
- We do a case analysis:
  - Case 1: $(r^2 > \varepsilon^3 \sigma^2)$: Apply additive approximation of Freize, Kannan and Vempala [FKV04].
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  - Case 2: $(r^2 \leq \varepsilon^3 \sigma^2)$: Lots of careful calculations!

# Possible extensions and applications

- Projective clustering: Fitting $k$, $j$-dimensional flats to a given dataset.
  - Note that fitting $k$, 0-dimensional flats is the classical $k$-means problem and fitting $k$, 1-dimensional flats is the $k$-lines problem.
  - Idea: Extending the sampling based ideas, being developed for $k$-means, to projective clustering.
  - Issues: Not clear if sampling helps in fitting even 1, 2-dimensional flat. The sampling based analysis breaks down even for fitting $k$, 1-dimensional flats (i.e., the $k$-lines problem).
- Streaming setting: Much better *sketching* based algorithms exist for rank-$k$ approximation.

# Observations and open problems

- Observation 1: A single length-squared sampled point gives a 2 factor approximation in expectation.
- Observation 2: Length-squared sampling does not work for rank-$k$ approximation for $k > 1$.
- Open question 1: Our multiplicative approximation uses $O(\frac{1}{\varepsilon^4})$ samples. Are these many samples necessary?
- Open question 2: Can we get multiplicative approximation using just adaptive length-squared sampling?

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Thank you