COL106: Data Structures and Algorithms

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Dynamic Programming

Problem

Let S and T be strings of characters. S is of length n and T is of length m. Find a *longest common subsequence* in S and T. This is a longest sequence of characters (not necessarily contiguous) that appear in both S and T.

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- Example $S =$ XYXZPQ, $T =$ YXQYXP
  - The longest common subsequence is XYXP
  - $S =$ XYXZPQ, $T =$ YXQYXP
- How do we define the subproblems?

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- What is $L(1, j)$ for $1 < j \leq m$?

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- Can we say something similar for $L(i, j)$ for $i, j \neq 1$?

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  - <u>Claim 1</u>: If $S[i] = T[j]$, then $L(i, j) = 1 + L(i-1, j-1)$.
  - <u>Claim 2</u>: If $S[i] \neq T[j]$, then $L(i, j) = \max\{L(i-1, j), L(i, j-1)\}$.

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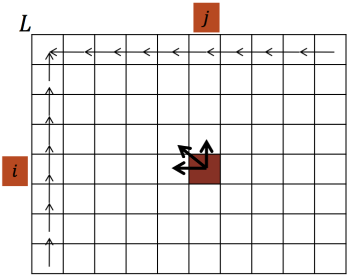

Figure: The arrows show the dependencies between subproblems.

Algorithm

`Length-LCS(S, T)`

- If $(S[1] = T[1])$, then $L[1,1] \leftarrow 1$ else $L[1,1] \leftarrow 0$
- For $j = 2$ to m
  - If $(S[1] = T[j])$, then $L[1,j] \leftarrow 1$ else $L[1,j] \leftarrow L[1,j-1]$
- For $i = 2$ to n
  - If $(S[i] = T[1])$, then $L[i,1] \leftarrow 1$ else $L[i,1] \leftarrow L[i-1,1]$
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  - For $j = 2$ to m
    - If $(S[i] = T[j])$ then $L[i, j] \leftarrow 1 + L[i - 1, j - 1]$
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- What is the running time of the above table-filling algorithm?

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- Return$(L[n,m])$

- What is the running time of the above table-filling algorithm?
  $O(nm)$

- How do we find a longest common subsequence?

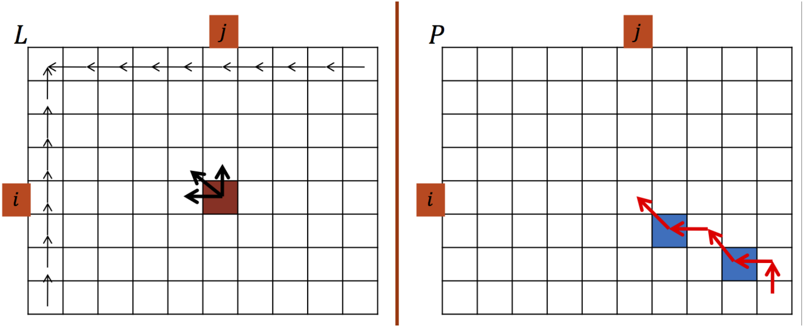


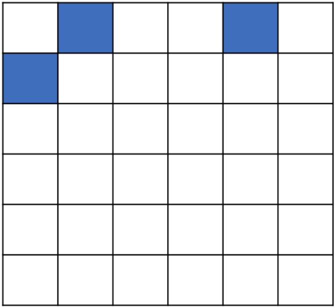
Figure: Array P is used to maintain the pointers to the appropriate subproblem. The blue squares give the position of the characters in a longest common subsequence.

- Example: $S = \text{XYXZPQ}$, $T = \text{YXQYXP}$

L

0	1	1	1	1	1
1					
1					
1					
1					
1					

P



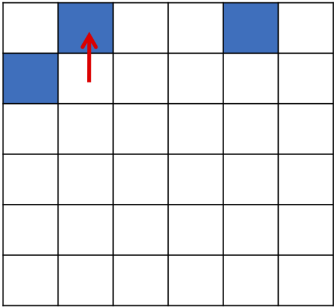
Dynamic Programming
Longest common subsequence

- Example: S = XYXZPQ, T = YXQYXP

L

0	1	1	1	1	1
1	1				
1					
1					
1					
1					

P



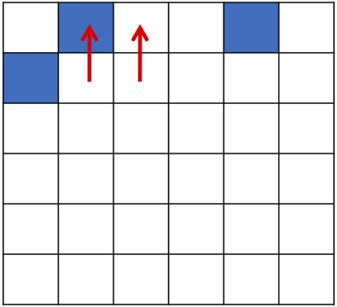
Dynamic Programming
Longest common subsequence

- Example: S = XYXZPQ, T = YXQYXP

L

0	1	1	1	1	1
1	1	1			
1					
1					
1					
1					

P

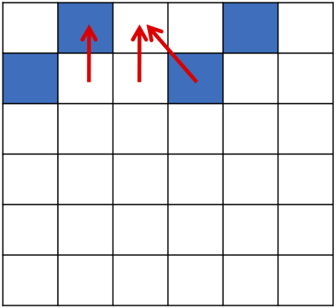


- Example: S = XYXZPQ, T = YXQYXP

L

0	1	1	1	1	1
1	1	1	2		
1					
1					
1					
1					

P



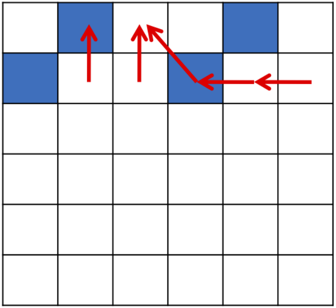
Dynamic Programming
Longest common subsequence

- Example: S = XYXZPQ, T = YXQYXP

L

0	1	1	1	1	1
1	1	1	2	2	2
1					
1					
1					
1					

P

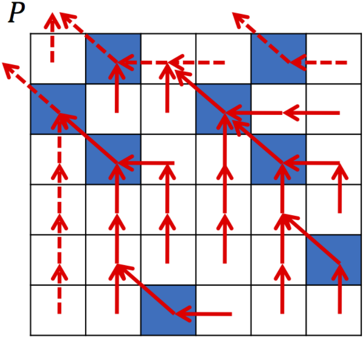


- Example: $S = $ XYXZPQ, $T = $ YXQYXP

L

0	1	1	1	1	1
1	1	1	2	2	2
1	2	2	2	3	3
1	2	2	2	3	3
1	2	2	2	3	4
1	2	3	3	3	4

P



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- Claim 1: If $i = 0$ or $j = 0$, then $L(i,j) = 0$.
- Claim 2: If $S[i] = T[j]$, then $L(i,j) = 1 + L(i-1, j-1)$.
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- Here is a simple recursive program to find the length of the longest common subsequence.

Algorithm

$\texttt{LCS-rec}(S, n, T, m)$
- If $(n = 0 \text{ OR } m = 0)$ then return(0)
- If $(S[n] = S[m])$ return$(1 + \texttt{LCS-rec}(S, n - 1, T, m - 1))$
- If $(S[n] \neq T[m])$ return$(\max\{\texttt{LCS-rec}(S, n, T, m - 1), \texttt{LCS-rec}(S, n - 1, T, m)\})$

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- What is the running time of this algorithm?

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- What is the running time of this algorithm?
  - This is exponentially large!

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- Here is a *memoized* version of the above algorithm.

Algorithm

`LCS-mem`(S, n, T, m)
- If $(n = 0$ OR $m = 0)$ then return(0)
- If ($L[n, m]$ is known) then return($L[n, m]$)
- If $(S[n] = S[m])$
  - $length \leftarrow 1 +$ `LCS-mem`$(S, n-1, T, m-1)$
- If $(S[n] \neq T[m])$
  - $length \leftarrow \max\{$`LCS-mem`$(S, n, T, m-1)$,
    `LCS-mem`$(S, n-1, T, m)\}$
- $L[n, m] \leftarrow length$
- return($length$)

- Here is a *memoized* version of the recursive algorithm.

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End