COL351: Analysis and Design of Algorithms

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Linear Programming: Rounding

Minimum Vertex Cover

Given a graph $G = (V, E)$ find a largest subset $S \subseteq V$ of vertices such that for every edge $(u, v) \in S$, at least one of u or v is in the set S.

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- For ever vertex $v \in V$, let x_v be an indicator variable (i.e., 0/1 variable) indicating whether x_v is in the vertex cover.
- Consider the following ILP:

$$\boxed{\begin{array}{l} \text{Minimize } \sum_{v \in V} x_v, \\ \text{Subject to } : \\ x_u + x_v \geq 1, \text{ for every edge } (u, v) \in E \\ x_v \in \{0, 1\}, \text{ for every vertex } v \in V \end{array}}$$

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- The issue is that ILP is NP-hard.
- Can we make use of the above ILP to get an approximate solution?

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- Claim 1: S is a vertex cover.
- Claim 2: Let OPT denote the size of the smallest vertex cover (and hence the optimal value of L). Then $|S| \leq 2 \cdot OPT$.

Weighted Set Cover (variant)

Given subsets $S_1, ..., S_m$ of a universe U of elements and positive weights $w_1, ..., w_m$ attached with these subsets. Find a subset S of $\{S_1, ..., S_m\}$ such that S covers all elements of U and $\sum_{i:S_i \in S} w_i$ is minimized. Also, assume that each element appears in at most f subsets.

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- How do we model this as an ILP?
- Let x_i denote an indicator variable for set S_i denoting whether S_i is taken.
- Consider the following ILP I:

$$\boxed{\begin{array}{l}\text{Minimize } \sum_{i=1}^{m} x_i \cdot w_i, \\ \text{Subject to } : \\ \sum_{i:e \in S_i} x_i \geq 1, \text{ for every element } e \in U \\ x_i \in \{0,1\}, \text{ for every } i \in \{1, ..., m\}\end{array}}$$

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- Let O_I denote the optimal value of the Integer LP and let O_R denote the optimal value of the relaxed LP.
- <u>Claim 1</u>: $O_I \geq O_R$.

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- Let O_I denote the optimal value of the Integer LP and let O_R denote the optimal value of the relaxed LP.
- Claim 1: $O_I \geq O_R$.
- Rounding:
  - Let $(r_1, ..., r_m)$ denote the optimal solution to R.
  - If $r_i \geq 1/f$, then put S_i in the set S.
- Claim 2: S is a set cover and $|S| \leq f \cdot OPT_I$, where OPT_I.

End